Solution of Problem 05-003 *by* TSZ HO CHAN[1] (University of Memphis).

The correct asymptotic behavior is given as follows.

THEOREM **1.** *For $x \geq 3$ (making sure that $\log\log x$ is positive),*

$$G(x) = \Theta\left(\frac{\log x}{\sqrt{\log\log x}}\right).$$

1. Preliminaries. First, we remind the reader that $f(x) = O(g(x))$ means that $|f(x)| \leq Cg(x)$ for some constant $C > 0$ and $f(x) = O_R(g(x))$ means that $|f(x)| \leq C_R g(x)$ for some constant $C_R > 0$ which may depend on R. Also $f(x) = \Theta(g(x))$ means that $f(x) = O(g(x))$ and $g(x) = O(f(x))$. We shall use Vinogradov's notation $f(x) \ll g(x)$, which is the same as $f(x) = O(g(x))$.

We also need some notation from number theory. First, $a|b$ means that a divides b, and $2^t||n$ means that n is divisible by 2^t but not 2^{t+1}. The arithmetic function $\omega(n)$ stands for the number of distinct prime factors of n. Also $\Omega(n)$ stands for the total number of prime factors of n (counting multiplicity), and $\overline{\Omega}(n)$ stands for the total number of odd prime factors of n (counting multiplicity). For example, $\omega(12) = 2$, $\Omega(12) = 3$, and $\overline{\Omega}(12) = 1$. Also, we use the convention $\omega(1) = \Omega(1) = \overline{\Omega}(1) = 0$.

Let us recall the discussion given by Marko Riedel. The divisor poset D_n of n is a product of chains. Suppose the prime factorization of n is

$$n = \prod_{j=1}^{m} p_j^{v_j}.$$

Then

$$D_n \cong \prod_{j=1}^{m} [v_j], \quad \text{where} \quad [v] \text{ is the chain } \{0, 1, \ldots, v\}.$$

This is an example of a PECK poset that has the Sperner property; i.e., its largest antichain is its largest rank level. The rank generating function for the above factorization is

$$\prod_{j=1}^{m} (1 + x + x^2 + \cdots + x^{v_j}). \tag{1}$$

In particular, $g(n)$ is the largest coefficient (or the middle coefficient) of this polynomial.

LEMMA **1.** *For any positive integer n such that $2^t||n$ for some integer $t \geq 0$,*

$$\binom{\omega(n)}{\lfloor \omega(n)/2 \rfloor} \leq g(n) \leq (t+1) \binom{\overline{\Omega}(n)}{\lfloor \overline{\Omega}(n)/2 \rfloor},$$

where $\lfloor x \rfloor$ stands for the greatest integer $\leq x$.

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Proof. The case when $n = 1$ is clear. So we assume that $n \geq 2$. For the lower bound, we notice that if $k|l$, then $g(k) \leq g(l)$ as the divisor poset of l contains the divisor poset of k. In particular, $g(\prod_{p|n} p) \leq g(n)$ where the product is over all the distinct prime factors p of n. However, for a squarefree integer like $\prod_{p|n} p$, the rank generating function is $\prod_{j=1}^{\omega(n)}(1+x)$. So

$$\binom{\omega(n)}{\lfloor \omega(n)/2 \rfloor} = g(\prod_{p|n} p) \leq g(n).$$

As for the upper bound, we need to introduce the following notation. Let $f(x) = \sum_{i=0}^{k} a_i x^i$ and $g(x) = \sum_{i=0}^{l} b_i x^i$ be two polynomials in $\mathbb{C}[x]$. We say $f(x) \prec g(x)$ if $|a_i| \leq b_i$ for all i. In particular, we must have that $k \leq l$ and $g(x)$ has nonnegative real coefficients. It is an easy exercise for the reader to check that if $f_1(x) \prec g_1(x)$ and $f_2(x) \prec g_2(x)$, then

$$f_1(x) + f_2(x) \prec g_1(x) + g_2(x) \text{ and } f_1(x)f_2(x) \prec g_1(x)g_2(x).$$

Thus, if $p_1 = 2$ and $v_1 = t$ in the prime factorization of n, the rank generating function in (1) is

$$(1 + x + x^2 + \cdots + x^t) \prod_{j=2}^{m} (1 + x + x^2 + \cdots + x^{v_j}) \tag{2}$$

$$\prec (1 + x + x^2 + \cdots + x^t) \prod_{j=2}^{m} (1 + x)^{v_j} = (1 + x + x^2 + \cdots + x^t)(1 + x)^{\overline{\Omega}(n)}.$$

The largest coefficient of $(1 + x)^{\overline{\Omega}(n)}$ is $\binom{\overline{\Omega}(n)}{\lfloor \overline{\Omega}(n)/2 \rfloor}$. Hence the largest coefficient in the right-hand side of (2) is $\leq (t + 1)\binom{\overline{\Omega}(n)}{\lfloor \overline{\Omega}(n)/2 \rfloor}$, and this gives the upper bound for $g(n)$. □

LEMMA **2.** *For any integer $k \geq 1$, the "middle" binomial coefficient*

$$\binom{k}{\lfloor k/2 \rfloor} = \Theta\left(\frac{2^k}{\sqrt{k}}\right).$$

Proof. We use Stirling's formula and leave it as an exercise for the reader. □

Our strategy is to use Lemmas 1 and 2 to obtain lower and upper bounds for $\sum_{n \leq x} g(n)$. An extra ingredient is the observation (Hardy–Ramanujan theorem [3]) that

$$\omega(n) \approx \Omega(n) \approx \overline{\Omega}(n) \approx \log\log x$$

for almost all integers $n \leq x$. Note that the upper bounds in Lemmas 4 and 6 below are much smaller than the main terms in Lemmas 5 and 7 with $z = 2$.

2. Proof of the lower bound. First, we need the following lemmas.

Lemma **3.** *For $x \geq 3$ and any integer $k \geq 1$,*

$$\sum_{\substack{n\leq x\\ \omega(n)=k}} 1 \leq C_1 \frac{x(\log\log x + C_2)^{k-1}}{(k-1)!\log x}$$

for some absolute constants $C_1, C_2 > 0$.

Proof. This is Hardy–Ramanujan inequality [3].[2] □

Lemma **4.** *For $x \geq 3$,*

$$\sum_{\substack{n\leq x\\ \omega(n)>6\log\log x}} 2^{\omega(n)} \ll \frac{x}{\log x}.$$

Proof. By Lemma 3 and Stirling's formula,

$$\sum_{\substack{n\leq x\\ \omega(n)>6\log\log x}} 2^{\omega(n)} = \sum_{6\log\log x<k\leq\frac{\log x}{\log 2}} \sum_{\substack{n\leq x\\ \omega(n)=k}} 2^k$$

$$\ll \sum_{k>6\log\log x} 2^k \frac{x(\log\log x + C_2)^{k-1}}{(k-1)!\log x} \ll \frac{x}{\log x} \sum_{k>6\log\log x} \frac{(2(\log\log x + C_2))^k}{k!}$$

$$\ll \frac{x}{\log x} \sum_{k>6\log\log x} \frac{(2(\log\log x + C_2))^k}{(k/e)^k} \ll \frac{x}{\log x} \sum_{k>6\log\log x} \left(\frac{2e}{6}\right)^k \ll \frac{x}{\log x},$$

as the geometric series converges. □

Lemma **5.** *Let $R > 0$ be fixed. Then uniformly for any complex number z with $|z| \leq R$,*

$$\sum_{n\leq x} z^{\omega(n)} = F(z)x(\log x)^{z-1} + O_R(x(\log x)^{Re\, z-2}),$$

where

$$F(z) = \frac{1}{\Gamma(z)} \prod_p \left(1 + \frac{z}{p-1}\right)\left(1 - \frac{1}{p}\right)^z$$

(the product is over all primes) and $\Gamma(z)$ is the gamma function.

Proof. See [4]. □

[2] Reference [3] can be found in *Collected Papers of Srinivasa Ramanujan*, G. H. Hardy, P. V. Sheshu Aiyar, and B. W. Wilson, eds., Cambridge University Press, 1927, pp. 262–275.

We are now ready to prove the lower bound. By Lemmas 1 and 2,

$$\sum_{n\le x} g(n) \ge \sum_{n\le x} \binom{\omega(n)}{\lfloor \omega(n)/2\rfloor} \gg \sum_{n\le x} \frac{2^{\omega(n)}}{\sqrt{\omega(n)}} \ge \frac{1}{\sqrt{6\log\log x}} \sum_{\substack{n\le x\\ \omega(n)\le 6\log\log x}} 2^{\omega(n)}. \tag{3}$$

By Lemmas 4 and 5 with $z=2$,

$$\sum_{\substack{n\le x\\ \omega(n)\le 6\log\log x}} 2^{\omega(n)} = \sum_{n\le x} 2^{\omega(n)} - \sum_{\substack{n\le x\\ \omega(n)> 6\log\log x}} 2^{\omega(n)} = Cx\log x + O(x)$$

for some constant $C>0$. Putting this into (3), we have

$$\sum_{n\le x} g(n) \gg \frac{x\log x}{\sqrt{\log\log x}},$$

which gives the lower bound in Theorem 1 after dividing both sides by x.

3. Proof of the upper bound. Again, we need some lemmas first.

Lemma **6.** *For $x\ge 3$,*

$$\sum_{\substack{n\le x\\ \overline{\Omega}(n)<\log\log x}} g(n) \ll x(\log x)^{\log 2}.$$

Proof. Suppose $n=2^t n'$, where $2^t || n$ and n' is odd. Note that $\overline{\Omega}(n)=\Omega(n')$. Then by Lemma 1,

$$\sum_{\substack{n\le x\\ \overline{\Omega}(n)<\log\log x}} g(n) \le \sum_{t\le \frac{\log x}{\log 2}} \sum_{\substack{n'\le \frac{x}{2^t}\\ \Omega(n')<\log\log x}} g(2^t n') \le \sum_{t\le \frac{\log x}{\log 2}} (t+1) \sum_{\substack{n'\le \frac{x}{2^t}\\ \Omega(n')<\log\log x}} \binom{\Omega(n')}{\lfloor \Omega(n')/2\rfloor}$$

$$\le \sum_{t=1}^{\infty} (t+1)\,\frac{x}{2^t}\,2^{\log\log x} \ll x(\log x)^{\log 2}.$$

□

Lemma **7.** *Let $\eta>0$ be fixed. Then uniformly for $|z|\le 3-\eta$,*

$$\sum_{n\le x} z^{\overline{\Omega}(n)} = G(z)x(\log x)^{z-1} + O_\eta(x(\log x)^{Re\, z-2}),$$

where

$$G(z) = \frac{2^{1-z}}{\Gamma(z)} \prod_{p\ge 3}\left(1-\frac{z}{p}\right)^{-1}\left(1-\frac{1}{p}\right)^{z}.$$

Proof. This is Lemma 7.4 in [2]. Its proof is similar to that of Lemma 5. □

Now we can embark on the proof of the upper bound. By Lemma 6 and a similar argument,

$$\sum_{n\le x} g(n) = \sum_{\substack{n\le x\\ \overline{\Omega}(n)<\log\log x}} g(n) + \sum_{\substack{n\le x\\ \overline{\Omega}(n)\ge\log\log x}} g(n) \tag{4}$$

$$\ll x(\log x)^{\log 2} + \sum_{t\le\frac{\log x}{\log 2}} \sum_{\substack{n'\le\frac{x}{2^t}\\ n'\mathrm{odd},\,\Omega(n')\ge\log\log x}} g(2^t n').$$

By Lemmas 1 and 2, the sum in the right-hand side of (4) does not exceed

$$\sum_{t\le\frac{\log x}{\log 2}} (t+1) \sum_{\substack{n'\le\frac{x}{2^t}\\ \overline{\Omega}(n')\ge\log\log x}} \binom{\overline{\Omega}(n')}{\lfloor\overline{\Omega}(n')/2\rfloor} \ll \sum_{t\le\frac{\log x}{\log 2}} (t+1) \sum_{n'\le\frac{x}{2^t}} \frac{2^{\overline{\Omega}(n')}}{\sqrt{\log\log x}} \tag{5}$$

$$\ll \frac{x\log x}{\sqrt{\log\log x}} \sum_{t=1}^{\infty} \frac{t+1}{2^t} \ll \frac{x\log x}{\sqrt{\log\log x}}$$

by Lemma 7 with $z=2$. Combining (4) and (5), we have

$$\sum_{n\le x} g(n) \ll \frac{x\log x}{\sqrt{\log\log x}},$$

which gives the upper bound of Theorem 1 after dividing both sides by x.

Acknowledgment. The author would like to thank Professor Cecil Rousseau for introducing him to this problem. Initial work on this problem was done while the author was at Central Michigan University.

Editorial note. While the above solution was being edited, the author found reference [1], in which it is asserted (p. 144) that

$$G(x) \sim \frac{\log x}{\sqrt{\pi\log\log x}}, \qquad x\to\infty.$$

REFERENCES

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[4] A. Selberg, *Note on a paper of L. G. Sathe*, J. Indian Math. Soc. (N.S.), 18 (1953), pp. 83–87.