Proof of a Figure Eight Volume Formula

In Problem 06-001, JONATHAN BORWEIN and MARC CHAMBERLAND asked for a self-contained proof of the figure eight volume formula

$$V_8 := \sum_{n=1}^{\infty} \frac{2\,\sin(n\pi/3)}{n^2} = \frac{4}{\sqrt{3}} \sum_{k=0}^{\infty} \left(-\frac{1}{3}\right)^k \frac{1+\frac{1}{3}\cdots+\frac{1}{2k+1}}{2k+1}.$$

Solution by VINICIUS-NICOLAE-PETRE ANGHEL[1] (AECL Chalk River Laboratory, Ontario, Canada).

This formula is a particular case of the relation

$$(1) \qquad \sum_{n=1}^{\infty} (2\sin\alpha)^n \frac{2\sin n\,(\pi/2-\alpha)}{n^2} = 4\tan\alpha \sum_{k=0}^{\infty} \left(-\left(\tan\alpha\right)^2\right)^k \frac{1+\frac{1}{3}+\ldots+\frac{1}{2k+1}}{2k+1},$$

which holds for $0 \le \alpha \le \pi/6$. The left-hand side of (1) can be expressed as

$$(2) \qquad \sum_{n=1}^{\infty} (2\sin\alpha)^n \frac{2\sin n\,(\pi/2-\alpha)}{n^2} = \frac{1}{i}\sum_{n=1}^{\infty} \frac{z^n}{n^2} - \frac{1}{i}\sum_{n=1}^{\infty} \frac{\bar{z}^n}{n^2},$$

where

$$z = 2\sin\alpha \exp\left(i\left(\pi/2-\alpha\right)\right) = 1 - e^{-2i\alpha},$$

and $\bar{z}$ is the complex conjugate of z. Because $|\alpha| \le \pi/6$, we have $|z| \le 1$, so the first of the two series that occur on the right-hand side of (2) converges to

$$F(z) := \frac{1}{i}\sum_{n=1}^{\infty} \frac{z^n}{n^2} = i\int_0^z \frac{\ln(1-u)}{u}\,du,$$

and similarly the second. (Note in passing that $iF(z)$ is the dilogarithm function $\mathrm{Li}_2(z)$.) The difference $F(z) - F(\bar{z})$ may be expressed as a complex integral along a straight line from $\bar{z}$ to 0 and then continuing along a straight line from 0 to z. The function $\ln(1-u)/u$ is analytic in the complex u-plane cut along the positive real axis from $u = 1$ to $+\infty$. Therefore, any contour not intersecting the branch cut and going from $\bar{z}$ to z will yield the same value for $F(z) - F(\overline{z})$. Let C be the contour

$$u(\phi) = 1 - e^{-i\phi}, \qquad -2\alpha \le \phi \le 2\alpha.$$

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Then the right-hand side of (2) becomes

$$F(z)-F(\overline{z}) = i\int_C \frac{\ln(1-u)}{u}\,du = i\int_{-2\alpha}^{2\alpha} \frac{-i\phi}{1-e^{-i\phi}}\,e^{-i\phi}\,i\,d\phi$$
$$= i\int_{-2\alpha}^{2\alpha} \frac{\phi\,e^{-i\phi/2}}{2i\sin(\phi/2)}\,d\phi = \int_0^{2\alpha} \frac{\phi}{\tan(\phi/2)}\,d\phi.$$

Substituting $\phi = 2\arctan t$, we obtain

$$F(z)-F(\overline{z}) = 4\int_0^{\tan\alpha} \frac{\arctan t}{t(1+t^2)}\,dt.$$

Expanding both $\arctan t$ and $1/(1+t^2)$ in series leads to

$$F(z)-F(\overline{z}) = 4\int_0^{\tan\alpha} \sum_{l=0}^{\infty} \frac{(-t^2)^l}{2l+1} \sum_{m=0}^{\infty} (-t^2)^m\,dt.$$

Series multiplication and term-by-term integration then produce

$$\sum_{n=1}^{\infty} (2\sin\alpha)^n \,\frac{2\sin n\,(\pi/2-\alpha)}{n^2} = F(z)-F(\overline{z}) \qquad (z = 1-e^{-2i\alpha})$$
$$= 4\int_0^{\tan\alpha} \sum_{k=0}^{\infty} \left(-t^2\right)^k \sum_{l=0}^{k} \frac{1}{2l+1}\,dt$$
$$= 4\tan\alpha \sum_{k=0}^{\infty} \frac{\left(-\tan^2\alpha\right)^k}{2k+1} \sum_{l=0}^{k} \frac{1}{2l+1},$$

which is (1). □

This problem has also been solved by S. Amghibech and the proposers.