A Subtle Integral?

Problem 06-002, *by* JONATHAN BORWEIN[1] (Dalhousie University, Halifax, NS, Canada) and MARC CHAMBERLAND[2] (Grinnell College, Grinnell, IA).

The following evaluation is given in [1] and used in several interesting applications. Below we present a self-contained version of the proof in [1] and ***we request a geometric proof***.

THEOREM **1.**

$$-\int_0^1 \frac{\log f(x)}{x}\,dx = \frac{\pi^2}{3ab}$$

for $0 < a < b$ *if* $f(x)^a - f(x)^b = x^a - x^b$ *and decreases.*

Note that, as Figure 1 illustrates, f is uniquely determined by this prescription.

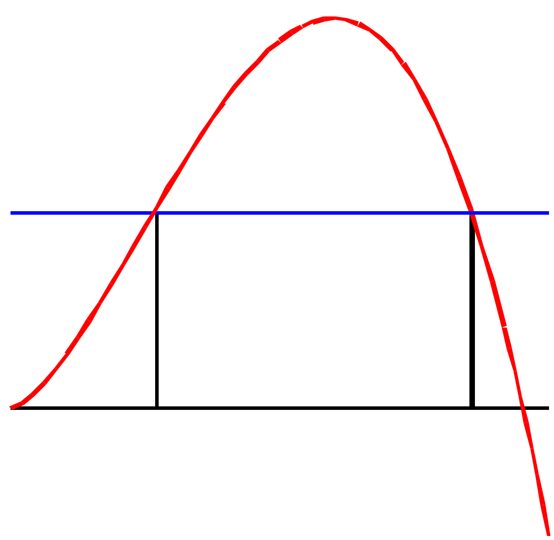

Figure 1: The graph of $x^a - x^b$.

Proof. One first observes that it suffices to consider the case $b := a+1$. The key step is the following identity asserting that for all $a > 0$

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$$G_a(1-x) := \sum_{n=1}^{\infty} \frac{\Gamma((a+1)n)}{\Gamma(an)} \frac{(x(1-x)^a)^n}{(na)\, n!} = -\log(1-x)$$

for $0 < x < 1/(a+1)$.

This is provided in Proposition 2 below.

First note that for all x

$$G_a(x) = G_a(f(x)) \tag{1}$$

and that $x^a - x^b$ has its maximum at $\mu := a/b = a/(a+1)$. Thus, we have the following:

1. $G_a(x) = -\log f(x)$ for $0 < x < \mu$.

2. $G_a(x) = -\log x$ for $\mu < x < 1$.

3. Using the β-function:

$$\begin{aligned} \int_0^1 \frac{G_a(x)}{x}\, dx &= \sum_{n=1}^{\infty} \frac{1}{an^2} \frac{\int_0^1 (1-x)^n x^{na-1}}{\beta(an,n)}\, dx \\ &= \sum_{n=1}^{\infty} \frac{1}{an^2} \frac{\beta(an,n+1)}{\beta(an,n)}\, dx = \frac{1}{a(a+1)} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6ab}. \end{aligned}$$

4. Clearly
$$\int_\mu^1 \frac{G_a(x)}{x}\, dx = -\int_\mu^1 \frac{\log(x)}{x}\, dx = \frac{1}{2} \log^2(\mu).$$

5. Moreover, integration by parts and (1) provide

$$\begin{aligned} -\int_0^\mu \frac{\log f(x)}{x}\, dx &= \int_0^\mu \frac{G_a(x)}{x}\, dx \\ &= -\int_\mu^1 G_a(x) \frac{f'(x)}{f(x)}\, dx = \int_\mu^1 \log x \frac{f'(x)}{f(x)}\, dx \\ &= (\log x \, \log f(x))|_\mu^1 - \int_\mu^1 \frac{\log f(x)}{x}\, dx \\ &= \log^2(\mu) - \int_\mu^1 \frac{\log f(x)}{x}\, dx. \end{aligned}$$

6. Hence,
$$\int_0^\mu \frac{-\log f(x)}{x}\, dx = \int_0^\mu \frac{G_a(x)}{x}\, dx = \frac{\pi^2}{6ab} - \frac{1}{2} \log^2(\mu),$$

   and
$$\int_\mu^1 \frac{-\log f(x)}{x}\, dx = \frac{\pi^2}{6ab} + \frac{1}{2} \log^2(\mu).$$

□

It remains to prove the following proposition.

PROPOSITION **2.**

$$G_a(1-x) := \sum_{n=1}^{\infty} \frac{\Gamma(an+n)}{\Gamma(an+1)} \frac{(x(1-x)^a)^n}{n!} = -\log(1-x) \tag{2}$$

for $0 < x < 1/(1+a)$, $a \geq 0$.

Proof. First we show that the series converges uniformly in x over the desired interval. The ratio test yields convergence if $ax(1-x)^a < 1$. It is easily shown that this function is maximized at $x = 1/(a+1)$ with a corresponding value less than one. The binomial series implies

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\Gamma(an+n)}{\Gamma(an+1)} \frac{(x(1-x)^a)^n}{n!} &= \sum_{n=1}^{\infty} \frac{\Gamma(an+n)}{\Gamma(an+1)} \frac{x^n}{n!} \sum_{m=0}^{\infty} \frac{(-1)^m x^m}{m!} (an)(an-1)\cdots(an-m+1) \\ &= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m (an+n-1)(an+n-2)\cdots(an+1-m)}{n!m!} x^{m+n}. \end{aligned}$$

Since

$$-\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n},$$

our result is equivalent to proving

$$\sigma_k := \sum_{n=1}^{k} \frac{(-1)^{k-n}(an+n-1)(an+n-2)\cdots(an+n-k+1)}{n!(k-n)!} = \frac{1}{k}$$

for $k = 1, 2, \ldots$. This is readily shown since

$$\begin{aligned} \sigma_k &= \frac{(-1)^k}{k!} \sum_{n=1}^{k} (-1)^n \binom{k}{n} \left(\frac{d^{k-1}}{x^{k-1}} x^{an+n-1} \right) \Bigg|_{x=1} \\ &= \frac{(-1)^k}{k!} \frac{d^{k-1}}{x^{k-1}} \left((1 - x^{an+n-1})^k - \frac{1}{x} \right) \Bigg|_{x=1} \\ &= \frac{(-1)^k}{k!} (-1)^k (k-1)! = \frac{1}{k}. \end{aligned}$$

□

Maple suggests the following in terms of hypergeometric functions.

CONJECTURE **3.** *Let a be a positive integer. Then*

$$G_a(1-x) = x(1-x)^a {}_{a+2}F_{a+1} \left[\begin{matrix} 1, 1, \frac{a+2}{a+1}, \frac{a+3}{a+1}, \ldots, \frac{2a+1}{a+1} \\ \frac{a+1}{a}, \frac{a+2}{a}, \ldots, \frac{2a-1}{a}, 2, 2 \end{matrix} ; \frac{(a+1)^{a+1}}{a^a} x(1-x)^a \right].$$

EXAMPLE **4.** *The function f can be determined explicitly in some cases:*

1. *for* $b = 2a : f(x) = (1 - x^a)^{1/a}$;
2. *for* $b = 3a : f(x) = (-x^a/2 + \sqrt{4 - 3x^{2a}})^{1/a}$;
3. *for* $2b = 3a : f(x) = (1/2 - x^a/2 + \sqrt{-3x^{2a} + 2x^a + 1}/2)^{1/a}$.

□

EXAMPLE **5.** *An interesting identity is produced if we let b approach a from above. First, let $f_{a,b}$ denote the unique decreasing function f which satisfies $f(x)^a - f(x)^b = x^a - x^b$. Then*

$$\begin{aligned} & f(x)^a - f(x)^b = x^a - x^b \\ \Leftrightarrow \quad & \int_a^b \frac{d}{dc} f_{a,b}^c(x) dc = \int_a^b \frac{d}{dc} x^c(x) dc \\ \Leftrightarrow \quad & \frac{\log f_{a,b}(x)}{\log x} = \frac{\int_a^b x^c(x) dc}{\int_a^b f_{a,b}^c(x) dc}. \end{aligned}$$

For simplicity, let $a = 1$. Letting $b \to 1$ gives

$$f_{1,1}(x)^{f_{1,1}(x)} = x^x.$$

Since the function $f(x) = x^x$ is decreasing on $(0, 1/e)$ and increasing on $(1/e, 1)$ and it satisfies $f(0) = f(1) = 1$, this shows the function $f_{1,1}(x)$ is well-defined and decreasing. It is straightforward to show that

$$f_{1,1}(x) = \frac{x \log x}{W(x \log x)}, \quad 0 \le x \le 1/e,$$

where W is the *Lambert W function* defined implicitly through $W(x)e^{W(x)} = x$. Using point 5 in the proof of Theorem 1 yields

$$-\int_0^{1/e} \frac{\log f_{1,1}(x)}{x} dx = -\int_0^{1/e} \frac{W(x \log x)}{x} dx = \frac{\pi^2}{6} - \frac{1}{2}.$$

□

EXAMPLE **6.** *The authors of* [1] *noted that the case $(a, b) = (1, 2)$ is simple since $f_{1,2}(x) = 1 - x$, leading to the well-known integral*

$$-\int_0^1 \frac{-\log(1-x)}{x} dx = \frac{\pi^2}{6}.$$

This example attempts to derive Theorem 1 as a special case of other integrals.

Let $g(x)$ be a smooth, increasing function satisfying $g(0) = 0$, and let $x = \bar{x}$ be the first positive value for which $g(x) = 1$. Define the function f such that

$$f(x) := (1 - g(x))^{xg'(x)/g(x)}. \tag{3}$$

Then

$$\begin{aligned} -\int_0^{\bar{x}} \frac{\log f(x)}{x} dx &= -\int_0^{\bar{x}} \frac{g'(x)}{g(x)} \log(1 - g(x)) dx \\ &= -\int_0^{\bar{x}} \log(1 - g(x)) d(\log g(x)) \\ &= -\int_0^1 \log(1 - u) d(\log u) \\ &= \frac{\pi^2}{6}. \end{aligned}$$

This means that to prove Theorem 1 it is sufficient to prove that there exists a function g, as described above, with $\bar{x} = 1$ for each function $f = f_{a,b}(x)^{ab/2}$. The case $f_{1,2}$ works with $g(x) = x$. The case $f_{a,2a}$ works with $g(x) = x^a$.

Note that the differential equation (3) can be solved in general to yield

$$g(x) = 1 - \text{dilog}^{-1}\left(\int_0^x \frac{-\log f(y)}{y} dy\right).$$

□

Example **7.** *Finally, we note that this will all work if we take a unimodal (analytic) function H with $H(0) = H(1) = 0$ and solve $H(x) = H(f(x))$. If we can find coefficients such that*

$$\sum_{n=1}^{\infty} A_n \frac{H(x)^n}{\int_0^1 H(x)^n} dx = -\log(x),$$

then we should be able to evaluate the corresponding integrals in terms of $\sum_{n>0} A_n$.

Status. This problem is open.

REFERENCE

[1] Alexander E. Holroyd, Thomas M. Liggett, and Dan Romik, Integrals, Partitions and Cellular Automata, *Trans. Amer. Math. Soc.* **356** (2004), 3349–3368. MSC (2000): Primary 26A06; Secondary 05A17, 60C05.