Boundary Conditions That Yield Conjugate Harmonic Functions

In Problem 06-004, David L. Russell[1] (Virginia Tech, Blacksburg, VA) posed the following question.

Let Ω be a bounded, simply connected domain in the plane with smooth boundary. Let u, v be harmonic functions in Ω, of class C^1 up to the boundary, and let g be a continuous function on $\partial\Omega$. Let $n = (n_1, n_2)$ be the exterior unit normal, and let $t = (-n_2, n_1)$ be the unit tangent vector. Assume that u and v satisfy the boundary conditions

$$\frac{\partial u}{\partial n} - \frac{\partial v}{\partial t} = g n_1,$$

$$\frac{\partial u}{\partial t} + \frac{\partial v}{\partial n} = g n_2.$$

Prove that $g = 0$ and $u + iv$ is analytic.

Solution by Michael Renardy[2] (Virginia Tech, Blacksburg, VA). Set $u + iv = \phi(z) + \psi(\bar{z})$. The boundary conditions can then be put in the form

$$2\psi'(\bar{z}) = g(n_1 + in_2)^2.$$

Next, let χ be the conformal mapping from the unit disk D to B. We have

$$n_1 + in_2 = q\chi'(\chi^{-1}(z))\chi^{-1}(z)$$

with q real. With

$$f(z) = (\chi'(\chi^{-1}(z))\chi^{-1}(z))^2,$$

and $gq^2 = p$, the boundary condition now assumes the form

$$2\psi'(\bar{z}) = pf(z).$$

We multiply by the conjugate of $f(z)$, and we find that the function

$$h(\bar{z}) = \psi'(\bar{z})\overline{f(z)}$$

takes real values on the boundary. Since h is an analytic function of $\bar{z}$, it follows that h is constant. This constant must be zero, because $f(\chi(0)) = 0$. Consequently, $\psi' = 0$.

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