A Liouville-Type Property for Differential Inequalities

Problem 06-005, *by* Vicenţiu Rădulescu[1] (Center of Nonlinear Analysis and Applications, University of Craiova, Craiova, Romania).

Consider real numbers p, q, r such that $q \geq 0$ and $r \geq 0$. Let f be a twice differentiable function in $(0, \infty)$ satisfying the differential inequality

$$\Big[x^2 f''(x) + pxf'(x) - q|f(x)|^{r-1} f(x)\Big] \operatorname{sgn} f(x) \geq 0 \qquad \text{for all } x > 0.$$

(a) Prove that if $q > 0$, then the following alternative holds: either f vanishes identically or there exists $A > 0$ such that both f and f' do not vanish in $[A, +\infty)$. Establish a corresponding result for $q = 0$.

(b) Consider $q > 0$ and let f be a nontrivial solution of the above differential inequality, such that f is positive and increasing in $[A, \infty)$. Prove that f is unbounded.

(c) Consider $q = 0$ and let f be a nontrivial solution of the above differential inequality, such that f is positive and increasing in $[A, \infty)$. Prove that if $p \leq 1$, then f is unbounded. Is the condition $p \leq 1$ necessary?

Remark. The Liouville theorem asserts that if f is a **bounded** twice differentiable function defined on the **whole** Euclidean space and such that $\Delta f = 0$, then f is constant. The result stated above establishes a **Liouville-type property** for **differential inequalities**. Indeed, if $q = 0$, then the associated differential **equation** is

$$f''(x) + p\,\frac{f'(x)}{x} = 0 \qquad \text{for all } x \in (0, \infty).$$

If $p = N - 1$, the above expression is that of the Laplace operator for functions with radial symmetry in $\mathbb{R}^N$. In contrast, our result asserts that the differential **inequality**

$$f''(x) + p\,\frac{f'(x)}{x} \geq 0 \qquad \text{for all } x \in (0, \infty)$$

admits nonconstant bounded solutions for all $p > 1$, but no bounded, positive and increasing (in a neighborhood of $+\infty$) solutions exist, provided $p \leq 1$. These extend some classical Liouville-type properties for subharmonic positive functions if $N = 1$ or $N = 2$.

Status. The proposer has a solution. Other solutions are welcome.

---

[1]E-mail: `vicentiu.radulescu@math.cnrs.fr`. Web: `http://www.inf.ucv.ro/~radulescu`.