A "Harmonic" Series

Problem 06-007, *by* Ovidiu Furdui[1] (Western Michigan University).

Let $H_n = \sum_{k=1}^{n} 1/k$ be the nth harmonic number. By elementary calculus,

$$\ln 3 = \int_n^{3n} \frac{dx}{x} > \sum_{k=n+1}^{3n} \frac{1}{k} = H_{3n} - H_n, \qquad n = 1, 2, 3, \dots ,$$

and the Euler–Maclaurin sum formula [1, p. 480] gives $H_n = \ln n + \gamma + \frac{1}{2n} + O(n^{-2})$, from which it follows that

$$\frac{\ln 3 - (H_{3n} - H_n)}{n} = \frac{1}{3n^2} + O(n^{-3}).$$

Hence $\sum_{n=1}^{\infty}(\ln 3 - (H_{3n} - H_n))/n$ is a convergent series with positive terms. Prove that

$$\sum_{n=1}^{\infty} \frac{\ln 3 - (H_{3n} - H_n)}{n} = \frac{5\pi^2}{36} - \frac{3\ln^2 3}{4}.$$

Also, it can be shown that

$$\sum_{n=1}^{\infty} \frac{\ln 2 - (H_{2n} - H_n)}{n} = \frac{\pi^2}{12} - \ln^2 2,$$

so it is natural to conjecture that for each $k \geq 2$,

$$\sum_{n=1}^{\infty} \frac{\ln k - (H_{kn} - H_n)}{n} = a_k\pi^2 + b_k \ln^2 k,$$

where a_k and b_k are rational numbers. Prove or disprove this conjecture.

Status. A solution is known. Others are welcome.

REFERENCE

[1] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, 2nd ed., Addison–Wesley, Boston, MA, 1994.

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