The Sum of the "Harmonic Series" Is Not Enough

In *Problem* 06-007, Ovidiu Furdui conjectured that for each integer $m \geq 2$, there are corresponding rational numbers a_m and b_m such that

$$S_m := \sum_{n=1}^{\infty} \frac{\log m - (H_{mn} - H_n)}{n} = a_m \pi^2 + b_m \log^2 m.$$

(Here $\log x$ is the natural logarithm of x.) The reader was asked to prove or disprove this conjecture.

Editorial note. The problem as stated is apparently quite hard. The "solutions" below determine S_m for all $m \geq 2$, but that does not settle the issue. For example, one finds that

$$S_5 = \frac{7\pi^2}{30} - \frac{5\log^2 5}{8} - \frac{1}{2}\log^2 \phi, \qquad \text{where} \qquad \phi = \frac{1+\sqrt{5}}{2},$$

which suggests that the conjecture is false. To be complete, however, a counterexample claim needs to include a proof that there do not exist rational numbers a and b such that

$$\log^2 \phi = a\pi^2 + b\log^2 5.$$

Neither the solvers nor the editors have such a proof.

Solution 1 *by* Omran Kouba[1] (Higher Institute for Applied Sciences and Technology, Damascus, Syria).

We will prove that for each $m \geq 2$,

$$(1) \quad \sum_{n=1}^{\infty} \frac{\log m - (H_{mn} - H_n)}{n} = \frac{(m-1)(m+2)}{24m}\pi^2 - \frac{1}{2}\log^2 m - \frac{1}{2}\sum_{j=1}^{m-1} \log^2\left(2\sin\frac{j\pi}{m}\right).$$

In particular,

$$S_2 = \frac{1}{12}\pi^2 - \log^2 2, \qquad S_3 = \frac{5}{36}\pi^2 - \frac{3}{4}\log^2 3, \qquad S_4 = \frac{3}{16}\pi^2 - \frac{11}{16}\log^2 4,$$

$$S_5 = \frac{7}{30}\pi^2 - \frac{5}{8}\log^2 5 - \frac{1}{2}\log^2\left(\frac{1+\sqrt{5}}{2}\right), \qquad S_6 = \frac{5}{18}\pi^2 - \frac{1}{2}\log^2 6 - \frac{1}{4}\log^2 3 - \frac{1}{2}\log^2 2.$$

This answers completely the question of evaluating the sums $(S_m)_{m \geq 2}$.

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For $m \geq 2$, let us define $Q_m(t) = 1 + t + \cdots + t^{m-1}$. Since $(1-t)Q_m(t) = 1 - t^m$ we have $(1-t)Q_m'(t) = Q_m(t) - mt^{m-1}$, and consequently, for $n \geq 1$ and $t \in (0,1)$,

$$\frac{Q_m'(t)}{Q_m(t)}(1 - t^{nm}) = (1-t)Q_m'(t)\frac{1-t^{nm}}{1-t^m} = \frac{1-t^{nm}}{1-t^m}(Q_m(t) - mt^{m-1}),$$

that is,

$$\begin{aligned}\frac{Q_m'(t)}{Q_m(t)}(1 - t^{nm}) &= \left(1 + t^m + t^{2m} + \cdots + t^{(n-1)m}\right)\left(1 + t + t^2 + \cdots + t^{m-1} - mt^{m-1}\right)\\ &= \sum_{j=1}^{nm} t^{j-1} - m\sum_{\ell=1}^{n} t^{m\ell-1}.\end{aligned}$$

We conclude that

$$\begin{aligned}\int_0^1 \frac{Q_m'(t)}{Q_m(t)}\, t^{nm}\, dt &= \int_0^1 \frac{Q_m'(t)}{Q_m(t)}\, dt - \sum_{j=1}^{nm}\int_0^1 t^{j-1}\, dt + m\sum_{\ell=1}^{n}\int_0^1 t^{m\ell-1}\, dt\\ &= \log m - H_{mn} + H_n.\end{aligned}$$

But the functions $\left(t \mapsto \frac{Q_m'(t)t^{mn}}{nQ_m(t)}\right)_{n\geq 1}$ are positive and continuous on $[0,1]$, so

$$\int_0^1 \frac{Q_m'(t)}{Q_m(t)}\left(\sum_{n=1}^{\infty}\frac{t^{mn}}{n}\right) dt = \sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 \frac{Q_m'(t)}{Q_m(t)} t^{mn}\, dt,$$

that is,

$$-\int_0^1 \frac{Q_m'(t)}{Q_m(t)}\log(1-t^m)\, dt = \sum_{n=1}^{\infty}\frac{\log m - (H_{mn} - H_n)}{n} = S_m.$$

Now $\log(1-t^m) = \log(1-t) + \log Q_m(t)$, so

$$\begin{aligned}S_m &= -\int_0^1 \frac{Q_m'(t)}{Q_m(t)}\log(1-t)\, dt - \int_0^1 \frac{Q_m'(t)}{Q_m(t)}\log(Q_m(t))\, dt\\ &= -\int_0^1 \frac{Q_m'(t)}{Q_m(t)}\log(1-t)\, dt - \left[\frac{1}{2}\log^2 Q_m(t)\right]_0^1.\end{aligned}$$

Finally,

$$S_m = -\frac{1}{2}\log^2 m + J_m \quad \text{with} \quad J_m = -\int_0^1 \frac{Q_m'(t)}{Q_k(t)}\log(1-t)\, dt, \tag{2}$$

and the problem is reduced to evaluating the integral J_m. Let ω denote the m^{th} root of unity: $e^{2\pi i/m}$. Then $Q_m(t) = \prod_{j=1}^{m-1}(t - \omega^j)$ and

$$\frac{Q'_m(t)}{Q_m(t)} = \sum_{j=1}^{m-1} \frac{1}{t - \omega^j}.$$

It follows that

$$J_m = \sum_{j=1}^{m-1} \int_0^1 \frac{\log(1-t)}{\omega^j - t}\, dt = \sum_{j=1}^{m-1} \int_0^1 \frac{\log(1-t)}{\omega^{m-j} - t}\, dt,$$

hence

$$2J_m = \sum_{j=1}^{m-1} \left(\int_0^1 \frac{\log(1-t)}{\omega^j - t}\, dt + \int_0^1 \frac{\log(1-t)}{\omega^{-j} - t}\, dt \right). \tag{3}$$

We will use the following lemma, the proof of which is postponed to the end.

Lemma. *Let $\Omega = \mathbb{C} \setminus [0, +\infty[$, that is, the set of complex numbers with a cut along the set of nonnegative real numbers. For z in Ω we define $F(z)$ by*

$$F(z) = \int_0^1 \frac{\log(1-t)}{z - t}\, dt.$$

Then F satisfies the functional equation

$$\forall\, z \in \Omega, \quad F(z) + F\left(\frac{1}{z}\right) = \frac{\pi^2}{6} - \operatorname{Log}(1-z)\ \operatorname{Log}\left(1 - \frac{1}{z}\right),$$

where Log *is the principal branch of the logarithm.*

With the notation of the lemma we can write (3) as follows:

$$\begin{aligned} 2J_m &= \sum_{j=1}^{m-1} \left(F(\omega^j) + F\left(\frac{1}{\omega^j}\right)\right) \\ &= \frac{(m-1)\pi^2}{6} - \sum_{j=1}^{m-1} \operatorname{Log}(1-\omega^j)\ \operatorname{Log}(1-\overline{\omega}^j) \\ &= \frac{(m-1)\pi^2}{6} - \sum_{j=1}^{m-1} |\operatorname{Log}(1-\omega^j)|^2. \end{aligned}$$

For $1 \le j < m$ we have $1 - \omega^j = 2\sin(j\pi/m)e^{i\pi/m - \pi/2}$, and consequently $\operatorname{Log}(1-\omega^j) = \log(2\sin(j\pi/m)) + i\pi\left(\frac{j}{m} - \frac{1}{2}\right)$. Therefore,

$$2J_m = \frac{(m-1)\pi^2}{6} - \sum_{j=1}^{m-1} \log^2\left(2\sin\frac{j\pi}{m}\right) - \pi^2 \sum_{j=1}^{m-1}\left(\frac{j}{m} - \frac{1}{2}\right)^2,$$

and

$$\begin{aligned}\sum_{j=1}^{m-1}\left(\frac{j}{m} - \frac{1}{2}\right)^2 &= \frac{1}{m^2}\cdot\frac{(m-1)m(2m-1)}{6} - \frac{1}{m}\cdot\frac{(m-1)m}{2} + \frac{m-1}{4}\\ &= \frac{(m-1)(m-2)}{12m},\end{aligned}$$

hence

$$J_m = \frac{(m-1)(m+2)\pi^2}{24m} - \frac{1}{2}\sum_{j=1}^{m-1}\log^2(2\sin(j\pi/m)). \tag{4}$$

Clearly (1) follows from (2) and (4). □

Proof of the lemma. Note first that both F and $z \mapsto \operatorname{Log}(1-z)$ are holomorphic in the connected region Ω, and since Ω is invariant under the holomorphic mapping $z \mapsto 1/z$, we conclude that

$$z \mapsto G(z) = F(z) + F\left(\frac{1}{z}\right) + \operatorname{Log}(1-z)\ \operatorname{Log}\left(1 - \frac{1}{z}\right)$$

is holomorphic in Ω, so to prove the lemma, we have to prove only that $G(x) = \pi^2/6$ for each negative real x. Now for $x \in (-\infty, 0)$ we have (integrating by parts)

$$\begin{aligned}F'(x) &= -\int_0^1 \frac{\log(1-t)}{(t-x)^2}\,dt = \left[\left(\frac{1}{t-x} - \frac{1}{1-x}\right)\log(1-t)\right]_0^1 + \frac{1}{1-x}\int_0^1 \frac{dt}{t-x}\\ &= \frac{1}{1-x}\log\left(1 - \frac{1}{x}\right),\end{aligned}$$

and consequently $G'(x) = 0$ for every negative real x. This proves that, for some constant c, we have $G(x) = c$ for all x in the interval $(-\infty, 0)$. Now, letting x tend to 0^-, and noting that $\lim_{x\to-\infty} F(x) = 0$, we conclude that

$$c = F(0) = \int_0^1 \frac{-\log(1-t)}{t}\,dt = \int_0^1 \sum_{n=1}^{\infty}\frac{t^{n-1}}{n}\,dt = \sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 t^{n-1}\,dt = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

This concludes the proof of the lemma. □

Solution 2 by THE EDITOR[2] (University of Memphis, Memphis, TN). In view of the asymptotic relation $H_n = \log n + \gamma + O(1/n)$, it is evident that the series defining S_m does indeed converge. Let

$$F(z) := \sum_{n=1}^{\infty} \frac{\log m - (H_{mn} - H_n)}{n}\, z^n.$$

Then F is analytic in the unit disc and $F(1) = S_m$. By Abel's Limit Theorem [1, p. 42], $S_m = \lim_{x\uparrow 1} F(x)$. It remains to find a suitable representation of F and calculate the required limit. Start with the well-known series[3]

$$\sum_{n=1}^{\infty} \frac{H_n}{n+1} z^{n+1} = \tfrac{1}{2} \operatorname{Log}^2(1-z), \qquad |z| < 1,$$

where $\operatorname{Log} z = \log|z| + i \operatorname{Arg} z$ denotes the principal branch of the logarithm. Then for $|z| < 1$,

$$G(z) := \sum_{n=1}^{\infty} \frac{H_n}{n} z^n = z + \sum_{n=1}^{\infty} \frac{H_n + 1/(n+1)}{n+1} z^{n+1} = \tfrac{1}{2} \operatorname{Log}^2(1-z) + \operatorname{Li}_2(z),$$

where $\operatorname{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}$ is the dilogarithm.

By the Multisection Formula [2, pp. 97–98], if $f(z) = \sum_{n=0}^{\infty} c_n\, z^n$, then

$$\mathcal{M}_m[f(z)] := \sum_{n=0}^{\infty} c_{mn}\, z^{mn} = \frac{1}{m} \sum_{k=0}^{m-1} f(\omega^k z), \qquad \omega = e^{2\pi i/m}.$$

In view of the relation

$$m \sum_{n=1}^{\infty} \frac{H_{mn}}{mn}\, z^n = m\mathcal{M}_m[G(\sqrt[m]{z})],$$

it follows that

$$F(z) = -\log m \operatorname{Log}(1-z) - m\mathcal{M}_m[G(\sqrt[m]{z})] + G(z), \qquad |z| < 1.$$

Split $F(z)$ into "continuous" and "singular" terms according to their behavior near $z = 1$. The contribution to S_m of a continuous term is simply its value at $z = 1$. For example, the dilogarithm terms in $G(z)$ and $-m\mathcal{M}_m[G(\sqrt[m]{z})]$ together contribute

$$\sum_{n=1}^{\infty} \frac{1}{n^2} - m \sum_{n=1}^{\infty} \frac{1}{(mn)^2} = \left(\frac{m-1}{m}\right) \operatorname{Li}_2(1) = \left(\frac{m-1}{m}\right) \frac{\pi^2}{6}.$$

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[3]Differentiate $\frac{1}{2} \operatorname{Log}^2(1-z)$ to get $\sum_{n=1}^{\infty} H_n z^n$ and then integrate.

The Multisection Formula gives

$$m\mathcal{M}_m\left[\tfrac{1}{2}\operatorname{Log}^2(1-\sqrt[m]{z})\right]=\tfrac{1}{2}\sum_{k=0}^{m-1}\operatorname{Log}^2\left(1-\omega^k\sqrt[m]{z}\right).$$

The $k=0$ term, namely $\frac{1}{2}\operatorname{Log}^2(1-\sqrt[m]{z})$, is singular and the remaining $m-1$ terms are continuous. The contribution to S_m from the latter is $-\frac{1}{2}\sum_{k=1}^{m-1}\operatorname{Log}^2(1-\omega^k)$.

It is necessary to calculate the limit of the sum of all singular terms as $x\uparrow 1$:

$$L=\lim_{x\uparrow 1}\left\{-\log m\log(1-x)+\tfrac{1}{2}\log^2(1-x)-\tfrac{1}{2}\log^2\left(1-\sqrt[m]{x}\right)\right\}.$$

To this end, set $x=1-e^{-u}$. Then $x\uparrow 1$ as $u\to\infty$ and

$$\begin{aligned}L&=\lim_{u\to\infty}\left\{u\log m+\tfrac{1}{2}u^2-\tfrac{1}{2}\log^2(1-(1-e^{-u})^{1/m}\right\}\\&=\lim_{u\to\infty}\left\{u\log m+\tfrac{1}{2}u^2-\tfrac{1}{2}\log^2\left(e^{-u}/m+O(e^{-2u})\right)\right\}\\&=\lim_{u\to\infty}\left\{u\log m+\tfrac{1}{2}u^2-\tfrac{1}{2}(u+\log m)^2+O(ue^{-u})\right\}\\&=\lim_{u\to\infty}\{-\tfrac{1}{2}\log^2 m+O(ue^{-u})\}=-\tfrac{1}{2}\log^2 m.\end{aligned}$$

Assembling all contributions, we have

$$S_m=-\frac{\log^2 m}{2}+\left(\frac{m-1}{m}\right)\frac{\pi^2}{6}-\frac{1}{2}\sum_{k=1}^{m-1}\operatorname{Log}^2(1-\omega^k). \tag{1}$$

To simplify the formula for S_m, note that the principal logarithm of $1-\omega^k$ is

$$\operatorname{Log}(1-\omega^k)=\log|1-\omega^k|+i\operatorname{Arg}(1-\omega^k)=\log\left(2\sin\frac{k\pi}{m}\right)+i\pi\left(\frac{k}{m}-\frac{1}{2}\right).$$

Hence

$$\begin{aligned}\sum_{k=1}^{m-1}\operatorname{Log}^2(1-\omega^k)=&\ \operatorname{Re}\sum_{k=1}^{m-1}\operatorname{Log}^2(1-\omega^k)\\&=\sum_{k=1}^{m-1}\log^2\left(2\sin\frac{k\pi}{m}\right)-\pi^2\sum_{k=1}^{m-1}\left(\frac{k}{m}-\frac{1}{2}\right)^2\\&=\sum_{k=1}^{m-1}\log^2\left(2\sin\frac{k\pi}{m}\right)-\pi^2\frac{(m-1)(m-2)}{12m}.\end{aligned}$$

Substitution of this result into (1) yields

$$S_m = \frac{(m-1)(m+2)}{24m}\,\pi^2 - \frac{1}{2}\log^2 m - \frac{1}{2}\sum_{k=1}^{m-1} \log^2\left(2\sin\frac{k\pi}{m}\right).$$

REFERENCES

[1] L. V. Ahlfors, *Complex Analysis*, McGraw-Hill, New York, 1966.

[2] Z. A. Melzak, *Companion to Concrete Mathematics*, John Wiley & Sons, New York, 1973.

Also solved by Thomas A. Dickens (Exxon Mobil Upstream Research Company, Houston, TX), *and* Nguyen Van Vinh *and* Ngo Phuoc Nguyen Ngoc (students) (Belarusian University, Minsk, Belarus).

Editorial note. The approach of Thomas A. Dickens was similar to that in *Solution* 1. In addition to formal analysis, Dickens used an integer-relation detection algorithm to study numerically the possibility of representing S_m by a linear combination of π^2 and $\log^2 m$ with rational coefficients. Using an integral representation of S_m, he applied the PSLQ algorithm to look for integers a, b, c such that $aS_m + b\pi^2 + c\log^2 m = 0$. Using 500-digit arithmetic, this method gave the known results for $m = 2, 3$, and 4 and found no solutions for $m \geq 5$. Nguyen Van Vinh and Ngo Phuoc Nguyen Ngoc first show that

$$S_m = \left(\frac{m-1}{m}\right)\frac{\pi^2}{6} - \frac{1}{m}\int_0^1 \left(\sum_{k=1}^{m-1} t^{k/m} - (m-1)\right)\frac{\log(1-t)}{t(1-t)}\,dt$$

and then use computer algebra to evaluate the integral for specific cases.