Fourier Transforms of Higher-Dimensional Analogues of the Hyperbolic Secant Function

Problem 07-001, *by* IAN MARTIN[1] (Harvard University).

The Fourier transform of the sech function is familiar. For the purposes of this problem, it is convenient to consider the scaled function $f(x) = [\text{sech}(x/2)]/2$. We have

$$\int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{e^{x/2}+e^{-x/2}}\,dx = \pi\ \text{sech}(\pi\omega). \tag{1}$$

For integer $n > 1$, the Fourier transform of $f(x)$ to the power n also takes a simple form. For example,

$$\int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{(e^{x/2}+e^{-x/2})^n}\,dx = g_n(\omega), \tag{2}$$

where the sequence of functions $g_n(\omega)$ begins:

$$\begin{aligned}
g_2(\omega) &= \pi\,\omega\,\text{cosech}(\pi\omega)\,,\\
g_3(\omega) &= \frac{\pi}{2!}\left(\omega^2+(1/2)^2\right)\text{sech}(\pi\omega)\,,\\
g_4(\omega) &= \frac{\pi}{3!}\,\omega\left(\omega^2+1^2\right)\text{cosech}(\pi\omega)\,,\\
g_5(\omega) &= \frac{\pi}{4!}\left(\omega^2+(1/2)^2\right)\left(\omega^2+(3/2)^2\right)\text{sech}(\pi\omega)\,,\\
g_6(\omega) &= \frac{\pi}{5!}\,\omega\left(\omega^2+1^2\right)\left(\omega^2+2^2\right)\text{cosech}(\pi\omega)\,,
\end{aligned}$$

as can be seen, in each case, by integrating around a rectangle in the complex plane with corners at $-R$, R, $R+\pi i$ and $-R+\pi i$, letting $R \to \infty$, and invoking the residue theorem.

These Fourier transforms crop up in the course of studying interactions between financial assets in a world with just two assets. If one wishes to extend the analysis to model interactions between assets in a world with three assets, the following analogue of (1) is of interest.

(a) What is the Fourier transform of

$$\left(e^{x/3}+e^{y/3}+e^{-(x+y)/3}\right)^{-1}\,? \tag{3}$$

(b) Similarly, the obvious analogue of (2) asks for the Fourier transform of

$$\left(e^{x/3}+e^{y/3}+e^{-(x+y)/3}\right)^{-n}\,. \tag{4}$$

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(c) More generally, it is of interest to find, for $m \geq 2$ and integer $n \geq 1$, the Fourier transform of

$$\left(e^{x_1/m} + \cdots + e^{x_{m-1}/m} + e^{-(x_1+\cdots+x_{m-1})/m}\right)^{-n} \tag{5}$$

and to investigate the properties of these Fourier transforms as m becomes large.

Status. The proposer has solutions of parts (a) and (b) and the Fourier transform for (c). Additional solutions are welcome.