## A Class of Higher-Dimensional Fourier Transforms

In Problem 07-001, Ian Martin[1] poses the following question. Assume that $m, n \in \mathbb{Z}^+$ and $m > 1$. Find the Fourier transform of

$$(e^{x_1/m} + \cdots + e^{x_{m-1}/m} + e^{-(x_1+x_2+\cdots+x_{m-1})/m})^{-n}.$$

These Fourier transforms crop up in the course of studying certain economic models.

*Solution* by the proposer.

We seek

(1)

$$I_m(\omega_1, \ldots, \omega_{m-1}) \equiv \int_{\mathbb{R}^{m-1}} \frac{e^{-ix_1\omega_1 - ix_2\omega_2 - \cdots - ix_{m-1}\omega_{m-1}}}{\left(e^{x_1/m} + \cdots + e^{x_{m-1}/m} + e^{-(x_1+x_2+\ldots+x_{m-1})/m}\right)^n} \, dx_1 \ldots dx_{m-1},$$

where $m$ and $n$ are positive integers and $m > 1$.

For notational convenience, write $x_m \equiv -x_1 - \cdots - x_{m-1}$, so $\sum_1^m x_i = 0$. For $i = 1, \ldots, m$, define

$$t_i = \frac{e^{x_i/m}}{e^{x_1/m} + \cdots + e^{x_m/m}}. \tag{2}$$

Note that the variables $t_i$ range between 0 and 1 (and, by construction, sum to 1) as the variables $\{x_i\}$ range around. Furthermore, we have

$$\prod_{k=1}^m t_k = \frac{e^{(x_1+\cdots+x_m)/m}}{\left(e^{x_1/m} + \cdots + e^{x_m/m}\right)^m} = \frac{1}{\left(e^{x_1/m} + \cdots + e^{x_m/m}\right)^m},$$

and

$$t_i^m = \frac{e^{x_i}}{\left(e^{x_1/m} + \cdots + e^{x_m/m}\right)^m},$$

so

$$e^{x_i} = \frac{t_i^m}{\prod_{k=1}^m t_k}, \qquad i = 1, \ldots, m. \tag{3}$$

Because of the linear dependence $\sum_{k=1}^m t_k = 1$, there are only $m-1$ independent variables and $t_m = 1 - t_1 - \cdots - t_{m-1}$, so we can rewrite

$$x_i = m \log t_i - \sum_{k=1}^{m-1} \log t_k - \log\left(1 - \sum_{k=1}^{m-1} t_k\right), \quad i = 1, \ldots, m-1. \tag{4}$$

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To make the change of variables specified in (2), we have to calculate the Jacobian

$$J \equiv \left| \frac{\partial(x_1, \ldots, x_{m-1})}{\partial(t_1, \ldots, t_{m-1})} \right| .$$

From (4),

$$\frac{\partial x_i}{\partial t_j} = \frac{1}{t_m} - \frac{1}{t_j} + \frac{m\delta_{ij}}{t_i} ,$$

where $\delta_{ij}$ equals one if $i = j$ and zero otherwise, so we can write

$$\begin{aligned}
\frac{\partial(x_1, \ldots, x_{m-1})}{\partial(t_1, \ldots, t_{m-1})} &= \begin{pmatrix} \frac{m}{t_1} & & & \\ & \frac{m}{t_2} & & \\ & & \ddots & \\ & & & \frac{m}{t_{m-1}} \end{pmatrix} + \begin{pmatrix} \frac{1}{t_m} - \frac{1}{t_1} & \frac{1}{t_m} - \frac{1}{t_2} & \cdots & \frac{1}{t_m} - \frac{1}{t_{m-1}} \\ \frac{1}{t_m} - \frac{1}{t_1} & \frac{1}{t_m} - \frac{1}{t_2} & \cdots & \frac{1}{t_m} - \frac{1}{t_{m-1}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{t_m} - \frac{1}{t_1} & \frac{1}{t_m} - \frac{1}{t_2} & \cdots & \frac{1}{t_m} - \frac{1}{t_{m-1}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{m}{t_1} & & & \\ & \frac{m}{t_2} & & \\ & & \ddots & \\ & & & \frac{m}{t_{m-1}} \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} \frac{1}{t_m} - \frac{1}{t_1} \\ \frac{1}{t_m} - \frac{1}{t_2} \\ \vdots \\ \frac{1}{t_m} - \frac{1}{t_{m-1}} \end{pmatrix}' \\
&\equiv \boldsymbol{A} + \boldsymbol{e}\boldsymbol{f}' .
\end{aligned}$$

The last line defines the $(m-1) \times (m-1)$ matrix $\boldsymbol{A}$ and the $(m-1)$-dimensional column vectors $\boldsymbol{e}$ and $\boldsymbol{f}$. The matrix $\boldsymbol{A}$ is diagonal: blanks indicate zeros. The prime symbol ($'$) denotes a transpose.

In order to calculate $J = \det(\boldsymbol{A} + \boldsymbol{e}\boldsymbol{f}')$ we can use the following result.

Fact **1 (matrix determinant lemma).** *Suppose that $\boldsymbol{A}$ is an invertible square matrix and that $\boldsymbol{e}$ and $\boldsymbol{f}$ are column vectors, each of length equal to the dimension of $\boldsymbol{A}$. Then*

$$\det(\boldsymbol{A} + \boldsymbol{e}\boldsymbol{f}') = \big(1 + \boldsymbol{f}'\boldsymbol{A}^{-1}\boldsymbol{e}\big) \det \boldsymbol{A} .$$

This fact is useful in the present case because $\boldsymbol{A}$ is diagonal, so its inverse and determinant are easily calculated. To be specific,

$$\det \boldsymbol{A} = \frac{m^{m-1}}{t_1 \cdots t_{m-1}} \qquad \text{and} \qquad \boldsymbol{A}^{-1} = \begin{pmatrix} \frac{t_1}{m} & & & \\ & \frac{t_2}{m} & & \\ & & \ddots & \\ & & & \frac{t_{m-1}}{m} \end{pmatrix} .$$

It follows that

$$J = \left[ 1 + \begin{pmatrix} \frac{1}{t_m} - \frac{1}{t_1} \\ \frac{1}{t_m} - \frac{1}{t_2} \\ \vdots \\ \frac{1}{t_m} - \frac{1}{t_{m-1}} \end{pmatrix}' \begin{pmatrix} \frac{t_1}{m} & & & \\ & \frac{t_2}{m} & & \\ & & \ddots & \\ & & & \frac{t_{m-1}}{m} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right] \frac{m^{m-1}}{t_1 \cdots t_{m-1}} = \frac{m^{m-2}}{t_1 \cdots t_m} .$$

We can now return to the integral $I_m$. For simplicity, write $\Pi$ for the product $\prod_{k=1}^m t_k$. Making the substitution suggested in (2),

$$\begin{aligned} I_m &= \int_{\mathbb{R}^{m-1}} \frac{(t_1^m/\Pi)^{-i\omega_1} (t_2^m/\Pi)^{-i\omega_2} \cdots (t_{m-1}^m/\Pi)^{-i\omega_{m-1}}}{(t_1 + t_2 + \cdots + t_m)^n \Pi^{-n/m}} \, J \, dt_1 \ldots dt_{m-1} \\ &= m^{m-2} \int_{\mathbb{R}^{m-1}} \Pi^{n/m} \left( \frac{t_1^m}{\Pi} \right)^{-i\omega_1} \left( \frac{t_2^m}{\Pi} \right)^{-im\omega_2} \cdots \left( \frac{t_{m-1}^m}{\Pi} \right)^{-i\omega_{m-1}} \frac{dt_1 \ldots dt_{m-1}}{t_1 \ldots t_{m-1} t_m} \\ &= m^{m-2} \int_{\mathbb{R}^{m-1}} \Big( t_1^{n/m + i\omega_1 + \cdots + i\omega_{m-1} - mi\omega_1} \cdot t_2^{n/m + i\omega_1 + \cdots + i\omega_{m-1} - mi\omega_2} \\ &\qquad \cdots t_{m-1}^{n/m + i\omega_1 + \cdots + i\omega_{m-1} - mi\omega_{m-1}} \cdot t_m^{n/m + i\omega_1 + \cdots + i\omega_{m-1}} \Big) \frac{dt_1 \ldots dt_{m-1}}{t_1 \ldots t_{m-1} t_m} . \end{aligned}$$

This is a Dirichlet integral of type 1, as discussed in [1, pp. 166–167]. As shown there, it can be evaluated in terms of $\Gamma$-functions: we have

$$I_m = \frac{m^{m-2}}{\Gamma(n)} \cdot \Gamma\left(n/m + i\omega_1 + i\omega_2 + \cdots + i\omega_{m-1}\right) \cdot \prod_{k=1}^{m-1} \Gamma\left(n/m + i\omega_1 + \cdots + i\omega_{m-1} - mi\omega_k\right) .$$

Defining $\mathcal{F}_n^m(\boldsymbol{\omega}) = I_m/(2\pi)^{m-1}$, where $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_{m-1})$, we have

(5)

$$\mathcal{F}_n^m(\boldsymbol{\omega}) = \frac{m^{m-2}}{(2\pi)^{m-1}} \cdot \frac{\Gamma\left(n/m + i\omega_1 + i\omega_2 + \cdots + i\omega_{m-1}\right)}{\Gamma(n)} \cdot \prod_{k=1}^{m-1} \Gamma\left(n/m + i\omega_1 + \cdots + i\omega_{m-1} - mi\omega_k\right) .$$

It follows from this definition of $\mathcal{F}_n^m(\boldsymbol{\omega})$, by the Fourier inversion theorem, that

$$\frac{1}{\left(e^{x_1/m} + e^{x_2/m} + \cdots + e^{-(x_1 + x_2 + \ldots + x_{m-1})/m}\right)^n} = \int_{\mathbb{R}^{m-1}} \mathcal{F}_n^m(\boldsymbol{\omega}) e^{i\boldsymbol{x} \cdot \boldsymbol{\omega}} \, d\boldsymbol{\omega} .$$

## REFERENCE


[1] J. Edwards, *A Treatise on the Integral Calculus, Vol.* II, Macmillan and Co., London, 1922. Available online from http://books.google.com/books?id=WB4PAAAAIAAJ.