A Multiple Integral in Terms of Stieltjes Constants

Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. In part (b) of *Problem* 07-002, Ovidiu Furdui asks for a proof that

$$I_m := \int_0^1 \cdots \int_0^1 \left\{ \frac{1}{x_1 x_2 \cdots x_m} \right\} dx_1 dx_2 \cdots dx_m = 1 - \sum_{k=0}^{m-1} \frac{\gamma_k}{k!}, \qquad m \geq 1, \tag{1}$$

where

$$\gamma_m = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{(\log k)^m}{k} - \int_1^n \frac{(\log x)^m}{x}\, dx \right) = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{(\log k)^m}{k} - \frac{(\log n)^{m+1}}{m+1} \right).$$

Note. The numbers $\gamma_0 = \gamma$ (Euler's constant), $\gamma_1, \gamma_2, \ldots$ are known as the *Stieltjes constants*. They occur in the Laurent expansion [1, p. 807]:

$$\zeta(s) = \frac{1}{s-1} + \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \gamma_m (s-1)^m.$$

REFERENCE

[1] M. Abramowitz and I. Stegun, eds., *Handbook of Mathematical Functions*, National Bureau of Standards, Washington, D.C., 1964.

Solution by Yaming Yu[1], University of California, Irvine. To simplify (1), make the change of variables

$$\begin{aligned} y_1 &= x_1^{-1}, \\ y_2 &= (x_1 x_2)^{-1}, \\ &\vdots \quad \vdots \\ y_m &= (x_1 \ldots x_m)^{-1}. \end{aligned}$$

Because $0 < x_i < 1$ for $i = 1, \ldots, m$ is equivalent to $1 < y_1 < \cdots < y_m$, we may rewrite (1) as

$$I_m = \int_1^\infty \int_1^{y_m} \cdots \int_1^{y_2} \frac{\{y_m\}}{y_1 \ldots y_{m-1} y_m^2}\, dy_1 \ldots dy_{m-1} dy_m,$$

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where $1/(y_1 \dots y_{m-1} y_m^2)$ is the Jacobian. Integrating out $y_1,\ y_2, \dots,\ y_{m-1}$ in that order leads to

$$I_m = \int_1^\infty \frac{\{y\} \log^{m-1} y}{(m-1)! y^2}\, dy, \qquad m \geq 1. \tag{2}$$

Integrating (2) by parts yields

$$\begin{aligned} m! I_m &= \lim_{n\to\infty} \sum_{k=1}^{n} \int_k^{k+1} \frac{m(y-k)\log^{m-1} y}{y^2}\, dy \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{(y-k)\log^m y}{y}\bigg|_k^{k+1} - \int_k^{k+1} \frac{k \log^m y}{y^2}\, dy \right) \\ &= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{\log^m (k+1)}{k+1} - \sum_{k=1}^{n} \int_k^{k+1} \frac{k \log^m y}{y^2}\, dy \right). \end{aligned}$$

On the other hand,

$$\begin{aligned} m! I_{m+1} &= \lim_{n\to\infty} \sum_{k=1}^{n} \int_k^{k+1} \frac{(y-k)\log^m y}{y^2}\, dy \\ &= \lim_{n\to\infty} \left(\int_1^{n+1} \frac{\log^m y}{y}\, dy - \sum_{k=1}^{n} \int_k^{k+1} \frac{k \log^m y}{y^2}\, dy \right) \\ &= \lim_{n\to\infty} \left(\frac{\log^{m+1}(n+1)}{m+1} - \sum_{k=1}^{n} \int_k^{k+1} \frac{k \log^m y}{y^2}\, dy \right). \end{aligned}$$

Thus, for $m \geq 1$,

$$\begin{aligned} m!(I_m - I_{m+1}) &= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{(\log(k+1))^m}{k+1} - \frac{(\log(n+1))^{m+1}}{m+1} \right) \\ &= \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{(\log k)^m}{k} - \frac{(\log n)^{m+1}}{m+1} \right) = \gamma_m, \end{aligned}$$

in view of the definition of the Stieltjes constant γ_m and the fact that the two middle expressions differ by $o(1)$. It is well known that $I_1 = 1 - \gamma$, where γ is Euler's constant. Hence

$$\sum_{k=1}^{m-1} (I_k - I_{k+1}) = \sum_{k=1}^{m-1} \frac{\gamma_k}{k!} \qquad \text{yields} \qquad I_1 - I_m = 1 - \gamma - I_m = \sum_{k=1}^{m-1} \frac{\gamma_k}{k!}.$$

Finally,

$$I_m = 1 - \sum_{k=0}^{m-1} \frac{\gamma_k}{k!}, \qquad m \geq 1,$$

which confirms the conjecture.

Remark. It is interesting to note that

$$\lim_{m\to\infty} I_m = 1 - \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} = -\zeta(0) = \frac{1}{2},$$

according to the series expansion of the Riemann zeta function $\zeta(s)$ around $s = 1$.

Also solved by VINICIUS NICOLAE PETRE ANGHEL (AECL Chalk River Laboratory, Ontario), LATISLAW MATEJÍČKA (Trenčín University of Alexander Dubček in Trenčín, Slovakia), and FRÉDÉRIC PEYSKENS (student, University of Ghent, Belgium). Partial solutions were contributed by KEVIN COULEMBIER (student, University of Ghent, Belgium) *and the proposer.*