Sums of Reciprocals of Polygonal Numbers and a Theorem of Gauss

In *Problem* 07-003, HONGWEI CHEN requests a general formula in terms of the digamma function for the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{p_r(n)}, \qquad r \ge 5,$$

where $p_r(n) = n[(r-2)n - (r-4)]/2$ is the *n*th *r*-sided polygonal number.

Solution by G. C. GREUBEL¹, Newport News, VA. Substitution yields

$$S_r = b \sum_{n=1}^{\infty} \frac{1}{n(n-a)},$$

where b = 2/(r-2) and a = (r-4)/(r-2). This sum can be evaluated using the following relation satisfied by the digamma function $\psi(z) := \Gamma'(z)/\Gamma(z)$:

$$\psi(z+1) + \gamma = \sum_{n=1}^{\infty} \frac{z}{n(n+z)}, \qquad z \neq -1, -2, -3, \dots$$

Note that $r \ge 5$ implies that a = (r-4)/(r-2) is not an integer. Hence setting z = -a leads to

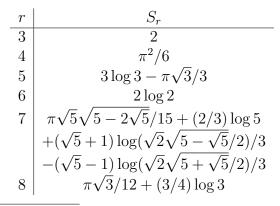
$$\psi(1-a) + \gamma = -a \sum_{n=1}^{\infty} \frac{1}{n(n-a)} = (-a) \frac{S_r}{b}$$

thus

$$\sum_{n=1}^{\infty} \frac{1}{p_r(n)} = S_r = -\frac{2}{r-4} \left[\gamma + \psi \left(\frac{2}{r-2} \right) \right]$$

is the desired formula. Exact values of S_r for $3 \le r \le 8$ are given in the table below.

Some Exact Values of S_r



¹E-mail: jthomae@gmail.com

Editorial note. Gauss proved that $\psi(x)$ can be expressed in terms of elementary functions when x is rational [1, p. 13]. Specifically, for 0 ,

$$\psi(p/q) = -\gamma - \frac{\pi}{2} \cot \frac{\pi p}{q} - \log q + 2\sum_{n=1}^{\lfloor q/2 \rfloor} \cos\left(\frac{2\pi n p}{q}\right) \log\left(2\sin\frac{\pi n}{q}\right),$$

where \sum' signifies that when q is even, the term q/2 is divided by 2. This gives rise to the following explicit formula for S_r in terms of $q = r - 2 \ge 3$:

$$S_r = \frac{1}{q-2} \left(\pi \cot\left(\frac{2\pi}{q}\right) + 2\log q - 4\sum_{n=1}^{\lfloor q/2 \rfloor} \cos\left(\frac{4\pi n}{q}\right) \log\left(2\sin\left(\frac{\pi n}{q}\right)\right) \right).$$

REFERENCE

[1] G. E. ANDREWS, R. ASKEY, AND R. ROY, *Special Functions*, Cambridge University Press, Cambridge, 1999.

Also solved by SAID AMGHIBECH (Quebec, Canada), KEVIN COULEMBIER (student, University of Ghent, Belgium), OVIDIU FURDUI (University of Toledo), FRÉDÉRIC PEYSKENS (student, University of Ghent, Belgium), SLAVKO SIMIC (Mathematical Institute SANU, Belgrade, Serbia), and the proposer.