No Finite Order Smoothness of f Implies $1/f$ Is Convex

In Problem 07-004, Slavko Simic raises the question of whether or not there is a finite integer n such that if f has smoothness order n, then $1/f$ is convex for sufficiently large x.

Solution by Peter Vandendriessche[1], Student, Ghent University, Belgium. We will show that the answer is negative: there is no finite n such that any function f of order n has $1/f(x)$ convex for sufficiently large x. To prove this, we will construct for arbitrary n a function $\phi(x)$ of order n and a sequence of points tending to infinity such that $1/\phi$ is not convex at these points.

It is a standard exercise to build a C^∞ function $r : \mathbb{R} \to \mathbb{R}$ such that r is nonnegative, r is identically zero outside the interval $[1/3, 2/3]$, and $\int_0^1 r(t)\,dt = 1$. Let $s_0(x) = \int_{-\infty}^x r(t)\,dt$ (equivalently the lower endpoint of the integral could be taken to be 0). Then s_0 is increasing, $s_0(x) = 0$ for $x \le 1/3$, and $s_0(x) = 1$ for $x \ge 2/3$. Define s_n inductively by $s_n(x) = n\int_{-\infty}^x s_{n-1}(t)\,dt$. Then s_n is C^∞, nonnegative, with its first n derivatives all nonnegative, and it satisfies

$$s_n(x) = 0 \quad \text{for} \quad x < 1/3 \quad \text{and} \quad (x-1)^n < s_n(x) < x^n \quad \text{for} \quad x \ge 1.$$

Note that $s_n(x) > (x-1)^n$ and $s_n'(x) = ns_{n-1}(x) < nx^{n-1}$. Hence we can choose $k > 1$ such that $s_n(k) > 4s_n'(k)$. Choose a constant $C > 1$ such that $2(C^2 + k - 1) > (C + k - 1)^2$ (for example $C = k^2$) and define

$$g(x) = e^{Cx} + (k-1)e^x.$$

Note that $g(0) = k$ and our assumption on C guarantees that $2g''(0) > (g'(0))^2$. Let $h(x) = s_n(g(x))$.

Clearly h is C^∞ since g and s_n are. Also, since s_n and g have their first n derivatives nonnegative, so does h. Moreover, for $x \ge 0$ we have $g(x) \ge k > 1$, so the derivatives of s_n at $g(x)$ are positive. Hence the first n derivatives of h are positive on $[0, \infty)$. Since $\lim_{x\to-\infty} g(x) = 0$ and $s_n(x) = 0$ for $x \le 1/3$, there is a constant $R > 0$ such that $h(x) = 0$ for $x \le -R$ (in fact $R = \log(3k)$ suffices). Finally, we compute

$$\begin{aligned} h(0)h''(0) &= s_n(k)[s_n'(k)g''(0) + s_n''(k)(g'(0))^2] \\ &> 4s_n'(k)[s_n'(k)g''(0)] \\ &> 2\left(s_n'(k)g'(0)\right)^2 = 2\left(h'(0)\right)^2. \end{aligned}$$

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For our final function we take $\phi(x) = \sum_{j=0}^{\infty} c_j h(x - jR)$, where $c_0 = 1$ and the remaining constants c_j will be determined below. Note that for any fixed $x \geq 0$, only finitely many terms in this are nonzero. Hence ϕ represents a C^{∞} function regardless of how rapidly the c_j grow. Since $h(x)$ has its first n derivatives nonnegative and positive for $x \geq 0$, we see that ϕ has its first n derivatives positive on $[0, \infty)$; that is, it is of order n. Finally, consider $\phi(mR)\phi''(mR) - 2(\phi'(mR))^2$ for some positive integer m. Terms in the sum for ϕ with $j > m$ contribute 0 to ϕ for all $x \leq mR$; hence this quantity depends only on $c_0, \ldots, c_m$. Further for $c_0, \ldots, c_{m-1}$ fixed it is a quadratic polynomial in c_m with leading coefficient $h(0)h''(0) - 2(h'(0))^2 > 0$. Hence, subsequently choosing c_m large enough (for $m = 1, 2, \ldots$) yields that

$$\phi(mR)\phi''(mR) - 2(\phi'(mR))^2 > 0$$

for all positive integers m. This inequality implies that $1/\phi(x)$ is concave at $x = mR$ for every positive integer m. Hence ϕ is our desired example.