A Natural Slow Series for e

Problem 08-002, *by* JONATHAN BORWEIN[1] (Dalhousie University, Halifax, NS, Canada).

1. Background. Typically numerical analysts are presented with slowly convergent series and the challenge is to find a good acceleration [2]. However, the discovery of (BBP) formulas for constants such as π makes it significant to ask: Are there natural slow series for a given constant κ? The formula

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right) \tag{1}$$

allows for individual hexadecimal bits to be computed [1, §3.4].

"Natural" and "slow" in this case require that the constant κ be expressible in the form

$$\kappa = \sum_{n=1}^{\infty} \frac{p(n)}{q(n)} \frac{1}{b^n}, \tag{2}$$

where p, q are integer polynomials of some fixed degree and $b \geq 1$ is an integer base. Many other constants with such natural and slow formulas (for $b \geq 2$) are listed in [1, §3.6].

A perusal of the literature uncovers very few series for e beyond Euler's

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} \tag{3}$$

and variations on this theme. Clearly the terms grow exponentially, not polynomially. The very efficacy of (3) has apparently limited the motivation to look for other series representations.

An unnatural but slow series for e [1, p. 328] is

$$e = 2 + \sum_{n=1}^{\infty} \frac{\left(n^2 + 2\,n\right)^{n+1} - n\left(n+1\right)^{2\,n+1}}{\left(n^2 + n\right)^{n+1}}, \tag{4}$$

which comes by rewriting $e = \lim_{n\to\infty}(1 + 1/n)^n$ as a series.

2. Questions. I pose the following questions.

1. Is there a slow natural series for e (even with $b = 1$)? I would conjecture not. Hence:
2. How much more natural a slow series for e can one find than the objectionable (4)?

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REFERENCES

[1] J. M. Borwein and D. H. Bailey, *Mathematics by Experiment: Plausible Reasoning in the 21st Century,* second expanded edition, A K Peters, Wellesley, MA, 2008.

[2] D. Levin, *Development of non-linear transformations for improving convergence of sequences*, Int. J. Comput. Math., B3 (1973), pp. 371–388.

Status. This problem is open.