Conjectured Bessel Moment Integrals

Problem 08-003, *by* David H. Bailey[1] (Lawrence Berkeley National Laboratory) and Jonathan Borwein[2] (Dalhousie University, Halifax, NS, Canada).

1. Background. A recent paper by the present authors, together with mathematical physicists David Broadhurst and M. Larry Glasser, explored Bessel moment integrals, namely definite integrals of the general form $\int_0^\infty t^m f^n(t)\,dt$, where the function $f(t)$ is one of the classical Bessel functions [2]. In that paper, numerous previously unknown analytic evaluations were obtained, using a combination of analytic methods together with some fairly high-powered numerical computations, often performed on highly parallel computers.

In several instances, while we were able to numerically discover what appears to be a solid analytic identity, based on extremely high-precision numerical computations, we were unable to find a rigorous proof. Thus we present here a brief list of some of these unproven but numerically confirmed identities. In the following, the functions $I_0(t)$ and $K_0(t)$ are the classical Bessel functions, as defined in [1, Chap. 15], while the function $\mathbf{K}(x)$ is the *complete elliptic integral* of the first kind, namely

$$\mathbf{K}(x) := \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - x^2 \sin^2 \phi}}.$$

These formulas also employ constants $K_3 := \mathbf{K}(k_3)$, $K_3' = \sqrt{3}K_3$, $K_{15} := \mathbf{K}(k_{15})$, $K_{5/3} = \mathbf{K}(k_{5/3})$, and C, where

$$k_3 = \frac{\sqrt{3}-1}{2\sqrt{2}} = \sin\left(\frac{\pi}{12}\right),$$

$$k_{15} = \frac{(2-\sqrt{3})(\sqrt{5}-\sqrt{3})(3-\sqrt{5})}{8\sqrt{2}},$$

$$k_{5/3} = \frac{(2-\sqrt{3})(\sqrt{5}+\sqrt{3})(3+\sqrt{5})}{8\sqrt{2}},$$

$$C := \frac{\pi}{16}\left(1 - \frac{1}{\sqrt{5}}\right)\left(1 + 2\sum_{n=1}^{\infty} \exp(-n^2\pi\sqrt{15})\right)^4.$$

Alternatively

$$C = \frac{\sqrt{5}-1}{4\sqrt{5}\pi} K_{15}^2 = \frac{1}{2\sqrt{15}\pi} K_{15} K_{5/3}.$$

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2. Conjectured Identities. Here are our selected conjectures. Can you find proofs for any (or all!) of these?

$$\frac{1}{\pi^2}\int_0^\infty tI_0(t)K_0^4(t)\,dt \stackrel{?}{=} C, \tag{1}$$

$$\frac{1}{\pi^2}\int_0^\infty t^3I_0(t)K_0^4(t)\,dt \stackrel{?}{=} \left(\frac{2}{15}\right)^2\left(13C-\frac{1}{10C}\right), \tag{2}$$

$$\frac{1}{\pi^2}\int_0^\infty t^5I_0(t)K_0^4(t)\,dt \stackrel{?}{=} \left(\frac{4}{15}\right)^3\left(43C-\frac{19}{40C}\right), \tag{3}$$

$$\frac{2}{\pi\sqrt{15}}\int_0^\infty tI_0^2(t)K_0^3(t)\,dt \stackrel{?}{=} C, \tag{4}$$

$$\frac{2}{\pi\sqrt{15}}\int_0^\infty t^3I_0^2(t)K_0^3(t)\,dt \stackrel{?}{=} \left(\frac{2}{15}\right)^2\left(13C+\frac{1}{10C}\right), \tag{5}$$

$$\frac{2}{\pi\sqrt{15}}\int_0^\infty t^5I_0^2(t)K_0^3(t)\,dt \stackrel{?}{=} \left(\frac{4}{15}\right)^3\left(43C+\frac{19}{40C}\right), \tag{6}$$

$$\int_0^\infty tI_0^2(t)K_0^2(t)K_0(2t)\,dt \stackrel{?}{=} \frac{1}{12}K_3K_3'. \tag{7}$$

A number of other related identities that are experimentally discovered but as yet unproven are mentioned in [2]. A discussion of the relative difficulty of each one on our list is discussed in [2].

REFERENCES

[1] Milton Abramowitz and Irene A. Stegun, *Handbook of Mathematical Functions*, Dover Publications, New York, 1965.

[2] David H. Bailey, Jonathan M. Borwein, David Broadhurst, and M. L. Glasser, *E*lliptic integral evaluations of Bessel moments and applications, J. Phys. A, 41 (2008), p. 205203; available at http://www.iop.org/EJ/article/1751-8121/41/20/205203/a8_20_205203.pdf.

Status. This problem is open.