Three Series for $1/\pi^2$ and $1/\pi^3$

Problem 08-004, *by* JONATHAN BORWEIN[1] (Dalhousie University, Halifax, NS, Canada).

1. Background. Now famous types of infinite series formulas were discovered by Ramanujan around 1910, but these were not well-known (nor fully proven) until quite recently when his writings were fully edited. Their proofs are based on elliptic integral or function theory and are described at length in [2]. One of these formulas is the remarkable

$$(1) \qquad \frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!\,(1103 + 26390k)}{(k!)^4\,396^{4k}}.$$

Each term of this series produces an additional *eight* correct digits in the result. Gosper used this formula to compute 17 million digits of π in 1985. At about the same time, David and Gregory Chudnovsky found the following variation of Ramanujan's formula:

$$(2) \qquad \frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k\,(6k)!\,(13591409 + 545140134k)}{(3k)!\,(k!)^3\,640320^{3k+3/2}}.$$

Each term of this series produces an additional 14 correct digits. They used this formula in several large calculations of π, culminating in 1994 with a calculation to over four billion decimal digits. In a related way, the Ramanujan-type series

$$(3) \qquad \frac{1}{\pi} = \sum_{n=0}^{\infty} \left(\frac{\binom{2n}{n}}{16^n} \right)^3 \frac{42\,n + 5}{16}$$

allows one to compute the billionth binary digit of $1/\pi$, or the like, *without computing the first half* of the series.

In some fairly recent papers, J. Guillera has exhibited several new Ramanujan-style series formulas for reciprocal powers of π, including the following [3, 5, 4]:

$$(4) \qquad \frac{128}{\pi^2} = \sum_{n=0}^{\infty} (-1)^n r(n)^5 (13 + 180n + 820n^2) \left(\frac{1}{32} \right)^{2n},$$

$$(5) \qquad \frac{8}{\pi^2} = \sum_{n=0}^{\infty} (-1)^n r(n)^5 (1 + 8n + 20n^2) \left(\frac{1}{2} \right)^{2n}, \quad \text{and}$$

$$(6) \qquad \frac{32}{\pi^3} = \sum_{n=0}^{\infty} r(n)^7 (1 + 14n + 76n^2 + 168n^3) \left(\frac{1}{8} \right)^{2n},$$

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where we define the function $r(n)$ as follows:

$$r(n) := \frac{(1/2)_n}{n!} = \frac{1/2 \cdot 3/2 \cdot \cdots \cdot (2n-1)/2}{n!} = \frac{\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+1)}.$$

Guillera proved (4) and (5) using an ingenious application of the Wilf–Zeilberger method [1, Chapter 3]. He ascribes series (6) to Gourevich,[2] who also found it using integer relation methods.

2. Two Problems. In [1, Chapters 2 & 3] we described a systematic hunt for other series of this form. We recovered all known identities and found no others. Guillera also provides other series for $1/\pi^2$ based on other Gamma function values—but here we restrict ourselves to $r(n)$.

No proof is known of (6), and the only proof known of (4) and (5) is, as alluded to, computational.

Problem 1. Prove (6).

Problem 2. Prove (4) and (5) by methods other than application of the Wilf–Zeilberger algorithm.

REFERENCES

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Status. This problem is open.

[2]In [1] we incorrectly recorded this series by writing $(1/32)^{2n}$ in place of $(1/8)^{2n}$.