A q-Series Identity

Problem 09-003, *by* Alexander E. Patkowski[1] (Centerville, MA).

Assume $|q| < 1$ and use the notation

$$(x)_n = (x;q)_n := \prod_{k=1}^{n} (1 - xq^{k-1}).$$

Show that

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n(-q)_n\, t^n}{(q)_n^3} = \frac{1}{(t)_\infty} \left(\sum_{n=0}^{\infty} \frac{t^n\, q^{n^2}}{(q)_n^2} \right)^2, \tag{1}$$

and use (1) to prove that

$$\sum_{n=0}^{\infty} \left(\frac{1}{(q)_\infty^3} - \frac{(q;q^2)_n(-q)_n}{(q)_n^3} \right) = \frac{1}{(q)_\infty^3} \left(\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{n(n+1)/2}}{1-q^n} \right). \tag{2}$$

Remark. I believe identity (1) is quite interesting in that it has special cases that have appeared in the literature. For example, setting $t = q$ gives the case $a = 1$ of [1, p. 389]:

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n(-q)_n q^n}{(q)_n^3} = \frac{1}{(q)_\infty^3} \left(\sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \right)^2. \tag{3}$$

Also, multiplying each side of (1) by $1 - t$ and taking the limit as $t \to 1-$ gives

$$\frac{1}{(q)_\infty^3} = \frac{1}{(q)_\infty} \left(\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n^2} \right)^2,$$

which is equivalent to the classical partition identity of Euler,

$$\frac{1}{(q)_\infty} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n^2}.$$

REFERENCE

[1] S. O. Warnaar, *Partial theta functions.* I. *Beyond the lost notebook*, Proc. London Math. Soc. (3), 87 (2003), pp. 363–395.

Status. A solution is known. Other solutions are welcome.

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