A Conjecture on the Modes of the Poisson Distribution of Order k

Problem 11-005, *by* A. N. Philippou[1] (Section of Statistics and Probability, University of Patras, Greece) and A. Saghafi (Iran University of Science & Technology, Tehran, Iran).

For any given positive integer k, let X be a random variable distributed as Poisson of order k with parameter λ; that is,

$$f_k(x;\lambda) = \sum_{x_1,\ldots,x_k} e^{-k\lambda} \frac{\lambda^{x_1+x_2+\cdots+x_k}}{x_1!x_2!\cdots x_k!},$$

where the summation is taken over all k-tuples of nonnegative integers $x_1, x_2, \ldots, x_k$ such that $x_1 + 2x_2 + \cdots + kx_k = x$ [1, 2, 3, 4].

Denote by $m_{k,\lambda}$ the mode(s) of $f_k(x;\lambda)$, and by $\lfloor u \rfloor$ the greatest integer not exceeding $u \in \mathbb{R}$. It is well known that $m_{1,\lambda} = \lfloor \lambda \rfloor$, when λ is not an integer, and $m_{1,\lambda} = \lambda - 1$ and λ, when λ is an integer. Show that

$$m_{k,\lambda} = \frac{k(k+1)}{2}\lambda - \left\lfloor \frac{k}{2} \right\rfloor, \qquad \text{for} \qquad k \geq 2, \quad \lambda \in \mathbb{N}.$$

This formula has been arrived at by numerical computation and a proof is desired. The problem of deriving the modes of $f_k(x;\lambda)$ was posed in [2, 3] and remains open. Luo [5] has shown the following sharp inequality: $m_{k,\lambda} \geq k\lambda\sqrt[k]{k!} - k(k+1)/2$.

Status. This problem is open.

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