Convex?

Problem 99-002, *by* Jonathan Borwein (CECM, Simon Fraser University, Burnaby, BC, Canada), Ian Affleck (CECM, Simon Fraser University), *and* Roland Girgensohn (GSF-Forshungszentrum, Institut für Biomathematik und Biometrie, Neuherberg, Germany).

For each $N \in \mathbb{N}$ define the function $f_N : \mathbb{R}_+^N \to \mathbb{R}_+$ by

$$f_N(x_1, \ldots, x_N) \ := \ \int_0^1 \left(1 - \prod_{i=1}^{N}(1 - t^{x_i})\right) \frac{dt}{t}.$$

For example,

$$\begin{aligned} f_1(x) &= \frac{1}{x}, \\ f_2(x_1, x_2) &= \frac{1}{x_1} + \frac{1}{x_2} - \frac{1}{x_1 + x_2}, \\ f_3(x_1, x_2, x_3) &= \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} - \frac{1}{x_1 + x_2} - \frac{1}{x_2 + x_3} - \frac{1}{x_1 + x_3} + \frac{1}{x_1 + x_2 + x_3}. \end{aligned}$$

(a) Show that f_N is positive and decreasing for every $N \in \mathbb{N}$.

(b) Show that f_N is convex for every $N \in \mathbb{N}$ or find a counterexample.

(c) Show that f_N is logarithmically convex[1] for every $N \in \mathbb{N}$ or find a counterexample.

(d) Show that $1/f_N$ is concave for every $N \in \mathbb{N}$ or find a counterexample.

Remarks by the proposers.

1. A complete answer is known only for (a). Moreover, it is easy to note that a positive answer to (d) implies a positive answer to (c), which in turn implies a positive answer to (b).

2. It is worth noting that

$$\lim_{x_N \to \infty} f_N(x_1, ..., x_{N-1}, x_N) = f_{N-1}(x_1, ..., x_{N-1}).$$

Thus, convexity properties of f_N are inherited by f_M for $M < N$.

[1] That is, $\log f_N$ is convex.

3. An explicit computation of the Hessian shows that $1/f_N$ is concave for $N < 5$. Indeed, each principal minor transpires to be a rational function with positive numerator and denominator. For $N = 4$ this requires a very large symbolic computation. Hence any counterexample lives in five or more dimensions.

4. For general N, it can be shown via the Hessian that every f_N is convex in a neighbourhood of $(1, 1, \ldots, 1)$. Moreover, differentiating the integral twice shows that f_N is decreasing and is convex with respect to each variable separately.

5. The function f_N arose—in highly disguised and more cumbersome form—as the objective function in a probabilistic network optimization problem. We consider a network *objective function* p_N given by

$$p_N(\vec{q}) = \sum_{\sigma \in S_N} \left(\prod_{i=1}^{N} \frac{q_{\sigma(i)}}{\sum_{j=i}^{N} q_{\sigma(j)}} \right) \left(\sum_{i=1}^{N} \frac{1}{\sum_{j=i}^{N} q_{\sigma(j)}} \right)$$

summed over *all* $N!$ permutations; so a typical term is

$$\left(\prod_{i=1}^{N} \frac{q_i}{\sum_{j=i}^{N} q_j} \right) \left(\sum_{i=1}^{N} \frac{1}{\sum_{j=i}^{N} q_j} \right).$$

For $N = 3$ this explicitly is

$$q_1 q_2 q_3 \left(\frac{1}{q_1 + q_2 + q_3} \right) \left(\frac{1}{q_2 + q_3} \right) \left(\frac{1}{q_3} \right) \left(\frac{1}{q_1 + q_2 + q_3} + \frac{1}{q_2 + q_3} + \frac{1}{q_3} \right).$$

Then it transpires that $p_N = f_N$, as was originally discovered within Maple via computation of the partial fraction decomposition.

6. A generalization of these functions is given by the following procedure: For a given function $g : I \to \mathbb{R}$ (here $I = \mathbb{R}_+$ or $I = \mathbb{R}$), define functions $f_N : I^N \to \mathbb{R}$ recursively by

$$f_1(x_1) := g(x_1),$$

$$\begin{aligned} f_{N+1}(x_1, \ldots, x_{N-1}, x_N, x_{N+1}) := f_N(x_1, \ldots, x_{N-1}, x_N) + f_N(x_1, \ldots, x_{N-1}, x_{N+1}) \\ - f_N(x_1, \ldots, x_{N-1}, x_N + x_{N+1}). \end{aligned}$$

For $g(x) = 1/x$ on $\mathbb{R}_+$, we get the functions discussed above, but for any given g, the same questions can be asked.

Status. As the proposers note above, only part (a) is completely solved.