Schedules in $\{0, 1\}^n$

Problem 99-003, *by* Dusan Jevtic (Applied Materials, Inc., Santa Clara, CA).

With $n > 1$, let $G_n(V, E)$ be the directed graph with vertex set $V = \{0, 1\}^n$ and edge set E given as follows:

$$(u, v) \in E \Leftrightarrow \begin{cases} u_1 = 0,\ v_1 = 1, & u_i = v_i\ (i \neq 1), \\ u_n = 1,\ v_n = 0, & u_i = v_i\ (i \neq n), \text{ or} \\ (u_k, u_{k+1}) = (1, 0), & \text{for some } k \in [1, n), \\ (v_k, v_{k+1}) = (0, 1), & u_i = v_i\ (i \neq k, k+1), \end{cases}$$

with $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$. A diagram of G_4 is shown in Figure 1. Let $f(n) = \sum_C \ell(C)$, where the sum is over all cycles in G_n and $\ell(C)$ denotes the length of C. Direct enumeration gives $f(2) = 6$, $f(3) = 24$, $f(4) = 180$, $f(5) = 29,112$, and $f(6) = 1,025,039,736$.

(a) Prove that the length of each cycle in G_n is a multiple of $n + 1$.

(b) Determine the distribution of cycle lengths in G_n. For example, all cycles in G_3 have length four, and in G_4 there are 24 cycles of length five and six cycles of length 10. As a first step, determine or estimate the length of the longest cycle in G_n.

(c) Obtain estimates or bounds for $f(n)$.

This problem arises in connection with multichamber tools for IC manufacturing. Wafers (silicon substrate) are processed in chambers and moved around (from one chamber to another, from a chamber to a load-lock, or from a load-lock to a chamber) by one or more robot arms. If there are n chambers, $C_1, C_2, \ldots, C_n$, then $u \in \{0, 1\}^n$ describes the occupancy of the chambers: if there is a wafer in chamber C_k, then $u_k = 1$; else $u_k = 0$. A *schedule* is a finite string $x \cdots uv \cdots x$ corresponding to a cycle in G_n that starts and ends at x. Hence $f(n)$ is the total number of schedules. If v is the immediate successor of u in the schedule, then u and v differ in at most two coordinates, and v must be chosen according to certain rules. These rules give rise to the requirements for $(u, v) \in E(G_n)$.

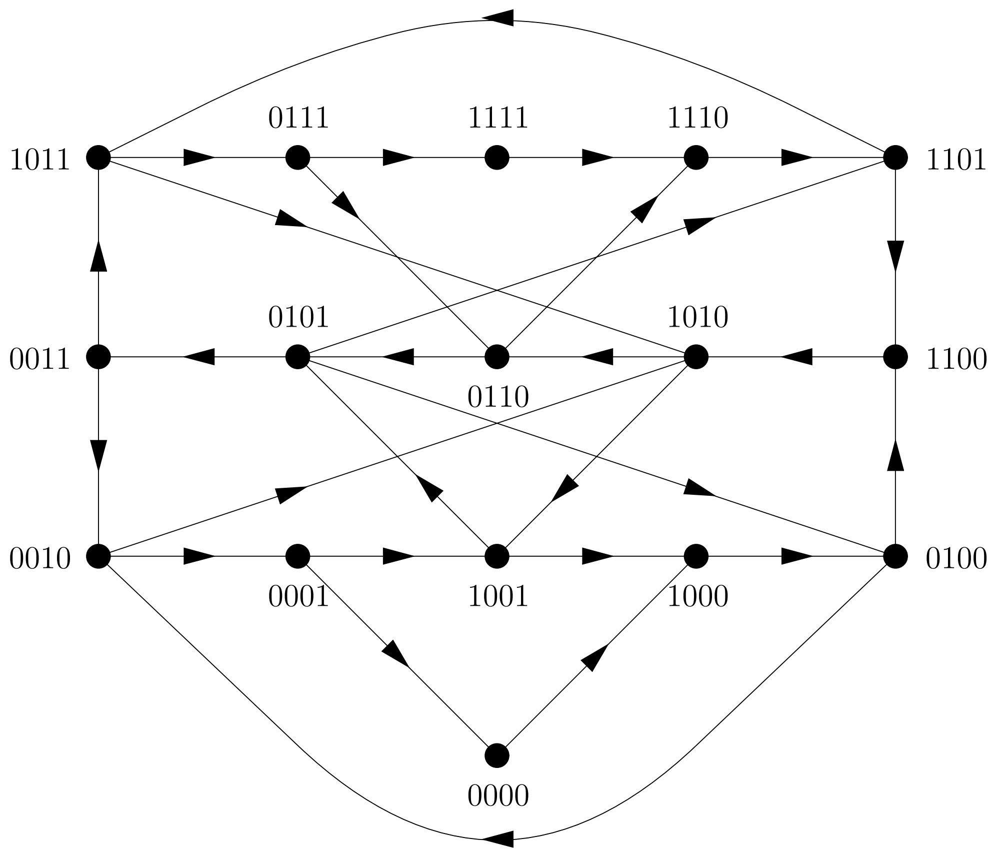

FIGURE 1. G_4

Status. A proof is known for part (a). Parts (b) and (c) are unsolved.