A Mellin Transform

Problem 99-004, *by* Allen R. Miller (Washington, DC).

If $\mu-\nu+\alpha-\beta$ is an odd integer N (positive or negative), the Mellin transform for $a>0$,

$$F(s) \equiv \int_0^\infty x^{s-1}\, {}_0F_1\left[\begin{matrix} -; \\ 1+\mu; \end{matrix} \quad -a^2x^2\right] {}_1F_2\left[\begin{matrix} \alpha; \\ \beta, 1+\nu; \end{matrix} \quad -a^2x^2\right] dx,$$

converges provided that

$$0< \text{ Re } s < \text{ Re } (2+\mu+\nu+\beta-\alpha) \qquad \text{and} \qquad 0< \text{ Re } s < \text{ Re } (3/2+\mu+2\alpha).$$

When $\alpha=\beta$, the latter conditional inequality is superfluous, $\mu-\nu=N$, $0<$ Re $s<$ Re $(2+\mu+\nu)$ for convergence, and the Mellin transform essentially reduces to the critical case of the Weber–Schafheitlin integral which is proportional to a ratio of products of gamma functions [1, p. 403]. For $\mu-\nu+\alpha-\beta=N$, the transform $F(s)$ has not been evaluated, but is believed to be obtainable essentially in terms of a finite number of generalized hypergeometric functions.

Find such a representation or show that none exists.

REFERENCE

[1] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, Cambridge, 1944.

Editorial Note. The above problem first appeared as Problem 97-13* in SIAM Review. Although a formal solution by C. C. Grosjean was published [*SIAM Rev.*, 40 (1998), pp. 726–729], the problem remains open, as its author points out in the following comment.

Comment by the proposer. We note that the solution provided by C. C. Grosjean is not, in general, a solution to the proposed problem. Indeed, Grosjean has formally shown, irrespective of any convergence criteria for the integral $F(s)$, that

$$F(s) = \frac{1}{2}a^{-s}\frac{\Gamma(\frac{s}{2})\Gamma(1+\mu)}{\Gamma(1+\mu-\frac{s}{2})}\, {}_3F_2\left[\begin{matrix} \alpha,\ \frac{s}{2},\ \frac{s}{2}-\mu; \\ \beta,\ 1+\nu; \end{matrix} \quad 1\right],$$

where $0<$ Re $s<$ Re $(\frac{3}{2}+\mu+2\alpha)$ and Re $s<$ Re $(1+\mu+\nu+\beta-\alpha)$, the latter of which is necessary also for the convergence of ${}_3F_2$ [1]. When $\mu-\nu+\alpha-\beta$ is an odd integer and $\alpha\neq 0$ or $\frac{s}{2}-\mu$ is a negative integer or zero, then ${}_3F_2$ [1] terminates so that the latter result does, in fact, (trivially) give a representation when $0 < \operatorname{Re} s < \operatorname{Re}(2+\mu+\nu+\beta-\alpha)$ which is necessary, in this case, only for the convergence of $F(s)$.

A rigorous analysis of the convergence criteria for $F(s)$ and a derivation of the latter result are given in [1]. Moreover, the proposed Problem 97-13* of finding a representation for $F(s)$ when $\mu-\nu+\alpha-\beta$ is an odd integer under the convergence conditions stated in the problem remains unsolved. This problem was motivated by the work in [1] and is also addressed in [2], where additionally it is shown that

$$F(s) = \frac{1}{2}a^{-s}\frac{\Gamma(\frac{s}{2})\Gamma(1+\mu)\Gamma(1+\nu+\beta-\alpha)\Gamma(1+\mu+\nu+\beta-\alpha-s)}{\Gamma(1+\mu-\frac{s}{2})\Gamma(1+\nu+\beta-\alpha-\frac{s}{2})\Gamma(1+\mu+\nu+\beta-\alpha-\frac{s}{2})}$$

$$\times\, {}_6F_5\left[\begin{matrix} 1+\frac{\nu+\beta-\alpha}{2},\ \nu+\beta-\alpha,\ \beta-\alpha,\ 1+\nu-\alpha,\ \frac{s}{2},\ \frac{s}{2}-\mu; \\ \frac{\nu+\beta-\alpha}{2},\ \beta,\ 1+\nu,\ 1+\nu+\beta-\alpha-\frac{s}{2},\ 1+\mu+\nu+\beta-\alpha-\frac{s}{2}; \end{matrix}\ -1\right],$$

where $0 < \operatorname{Re} s < \operatorname{Re}(2+\mu+\nu+\beta-\alpha)$ and $0 < \operatorname{Re} s < \operatorname{Re}(\mu+2\alpha)$. This representation also falls short of the desired result since the problem demands that $0 < \operatorname{Re} s < \operatorname{Re}(\frac{3}{2}+\mu+2\alpha)$. We mention also that a generalization of the proposed problem, essentially with ${}_1F_2[-a^2x^2]$ replaced by ${}_pF_{p+1}[-a^2x^2]$, is discussed in [3].

REFERENCES

[1] A. R. Miller and H. M. Srivastava, *On the Mellin transform of a product of hypergeometric functions*, J. Austral. Math. Soc. Ser. B, 40 (1998), pp. 222–237.

[2] A. R. Miller, *On the critical case of the Weber-Schafheitlin integral and a certain generalization*, J. Comput. Appl. Math., Special Issue on Higher Transcendental Functions and Applications, to appear.

[3] A. R. Miller, *On the Mellin transform of products of Bessel and generalized hypergeometric functions*, J. Comput. Appl. Math., 85 (1997), pp. 271–286.

Status. This problem is unsolved.