Convex! II

Solution of Problem 99-002 *by* Jonathan Borwein (CECM, Simon Fraser University, Burnaby, BC, V5A 1S6, Canada (jborwein@cecm.sfu.ca) *and* Omar Hijab (Temple University, Philadelphia, PA 19122 (hijab@math.temple.edu).

With f_N as in the problem statement, we show that f_N and $\log f_N$ are convex and $1/f_N$ is concave on the positive orthant $\mathbf{R}_+^N$. The first step is to display f_N as an expectation.

Lemma **1.** *If* $x = (x_1, \ldots, x_N)$, *we have*

$$f_N(x) := \int_0^1 \left(1 - \prod_{i=1}^N (1 - t^{x_i})\right) \frac{dt}{t} = E\left(\max\left(\frac{X_1}{x_1}, \ldots, \frac{X_N}{x_N}\right)\right),$$

where X_i *are independent positive random variables satisfying* $P(X_i > t) = e^{-t}$, $i = 1, \ldots, N$.

Proof. By independence we have

$$f_N(x) = \int_0^\infty \left(1 - \prod_{i=1}^N (1 - e^{-tx_i})\right) dt = \int_0^\infty P\left(\max\left(\frac{X_1}{x_1}, \ldots, \frac{X_N}{x_N}\right) > t\right) dt.$$

Since $\int_0^\infty P(X > t)\, dt = E(X)$ for any positive random variable X, the result follows. □

Then from this representation it is immediate that f_N is positive, decreasing, and strictly convex. To derive the stronger result that $1/f_N$ is concave, we will need an additional lemma.

Lemma **2.** *Let* $X : \mathbf{R}_+^N \times \Omega \to (0, \infty)$ *be a family of positive random variables such that the sample functions* $X(\cdot, \omega)$ *are concave for all* ω. *Then*

$$f(x) := E\left(\frac{1}{X(x)}\right)$$

implies $1/f$ *is concave and hence* $\log f$ *and* f *are convex.*

Proof. Let $h(a, b) = 2ab/(a + b)$. Then h is concave and concavity of $1/f$ is equivalent to

$$f((x + x')/2) \leq h\left(f(x), f(x')\right). \tag{1}$$

To establish this, use the concavity of the sample functions and Jensen's inequality applied to h to obtain

$$f((x + x')/2) \leq E\left(\frac{1}{(X(x) + X(x'))/2}\right) = E\left(h\left(\frac{1}{X(x)}, \frac{1}{X(x')}\right)\right) \leq h\left(f(x), f(x')\right).$$

Since $h(a, b) \leq \sqrt{ab} \leq (a + b)/2$, the log-convexity and the convexity of f follow from (1). This completes the proof. □

Now we can complete the solution by setting in Lemma 2

$$X(x) = \min\left(\frac{x_1}{X_1}, \dots, \frac{x_N}{X_N}\right),$$

where X_i are as in Lemma 1, obtaining the concavity of $1/f_N$, the log-convexity of f_N, and the convexity of f_N.

REMARK **1.** *The same techniques can be used to derive the strict convexity, log-convexity, and inverse-concavity of*

$$\int_0^1 \left(1 - \prod_{i=1}^{N} (1 - t^{g(x_i)})\right) \frac{dt}{t}$$

for g concave.

REMARK **2.** *Since $P(X_i \le t) = 1 - e^{-t}$, the density of X_i equals $P(t < X_i \le t + dt) = e^{-t}dt$. Since the X_i's are independent, the density of $(X_1, \dots, X_N)$ is the product of $e^{-t_i}dt_i$, $i = 1, \dots, N$. Substituting $t_i = y_i x_i$, Lemma 1 expresses f_N as a Laplace transform*

$$f_N(x) = \int_{\mathbf{R}_+^N} e^{-\langle x, y\rangle} \left(\prod_{i=1}^{N} x_i\right) \max(y_1, \dots, y_N)\, dy$$

(where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product).

REFERENCES

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