Partial Quicksort *

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Abstract

This short note considers the following common problem: rearrange a given array with n elements, so that the first m places contain the m smallest elements in ascending order. We propose here a simple variation of quicksort that efficiently solves the problem, and show and quantify how it outperforms other common alternatives.

1 Introduction

In many applications, we need to obtain a sorted list of the m smallest elements of a given set of n elements. This problem is known as *partial sorting*. Sorting the whole array is an obvious solution, but it clearly does more work than necessary.

A usual solution to this problem is to make a heap with the n given elements (in linear time) and then perform m successive extractions of the minimum element. Historically, this has been the way in which C++ STL's `partial_sort` function has been implemented [5]. Its most prominent feature is that it guarantees $\Theta(n + m \log n)$ worst-case performance.

Another solution, begins by building a max-heap with the first m elements of the given array, then scanning the remaining $n - m$ elements and updating the heap as necessary, so that at any given moment the heap contains the m smallest elements seen so far. Finally, the heap is sorted. Its worst-case cost is $\Theta((m+n) \log m)$ and it is not an interesting alternative unless m is quite small or we have to process the input on-line.

Last but not least, we can solve the problem by first using a selection algorithm to find the mth smallest element. Most selection algorithms also rearrange the array so that the elements which are smaller than the sought element are to its left, the sought element is at the mth component, and the elements which are larger are to the right. Then, after the mth smallest element has been found, we finish the algorithm sorting the subarray to the left of the mth element. Using efficient algorithms for both tasks (selection and sort) the total cost is $\Theta(n + m \log m)$. The obvious choice is to use Hoare's quickselect and quicksort algorithms [3, 4]. Then, the cost stated above is only guaranteed on average, but in practice this combination should outperform most other choices. The Copenhagen STL group implements `partial_sort` that way—actually, using finely tuned, highly optimized variants of these algorithms (http://www.cphstl.dk). For convenience, we call this combination of the two algorithms *quickselsort*.

In this paper we propose *partial quicksort*, a simple and elegant variant of quicksort that solves the partial sorting problem, by combining selection and sorting into a single algorithm. To the best of the author's knowledge the algorithm has not been formally proposed before. However, because of its simplicity, it may have been around for many years.

Partial quicksort has the same asymptotic average complexity as quickselsort, namely, $\Theta(n + m \log m)$, but does less work. In particular, if we consider the standard variants of quickselsort and partial quicksort, the latter saves $2m - 4 \ln m + \mathcal{O}(1)$ comparisons and $m/3 - 5/6 \ln m + \mathcal{O}(1)$ exchanges.

The rest of this short note is devoted to present the algorithm and to analyze the average number of comparisons and exchanges of the basic variant. Finally, we compare them with the corresponding values for the quickselsort algorithm.

2 The algorithm

Partial quicksort works as follows. In a given recursive call we receive the value m and a subarray $A[i..j]$ such that $i \leq m$. We must rearrange the subarray so that $A[i..m]$ contains the $m - i + 1$ smallest elements of $A[i..j]$ in ascending order; if $m > j$ that means that we must fully sort $A[i..j]$. The initial recursive call is with $A[1..n]$.

If the array contains one or no elements, then we are done. Otherwise, one of its elements is chosen as the pivot, say p, and the array $A[i..j]$ is partitioned so that $A[i..k-1]$ contains the elements that are smaller

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than p, $A[k]$ contains the pivot and $A[k+1..j]$ contains the elements which are larger than the pivot[1]. We will assume that exactly $n-1$ element comparisons are necessary to carry out the partitioning of the array and that partitioning preserves randomness [16]. If $k \geq m$ then all the sought elements are in the left subarray, so we only need to make a recursive call on that subarray. Otherwise, if $k < m$ then we have to make a recursive call on both subarrays. Notice that when $k \leq m$ the function will behave exactly as quicksort in the left subarray and fully sort those elements, whereas the recursive call on the right subarray has still the job to put the mth element into its final place.

Algorithm 1 depicts the basic or standard variant of the algorithm. Usual optimizations like recursion cutoff, sampling for pivot selection, recursion removal, optimized partition loops [1, 14, 15, 16], etc. can be applied here and presumably yield benefits similar to those for the quicksort and quickselect algorithms. However, we do not analyze these refined variants in this paper and we stick to the simpler case.

```
void partial_quicksort(vector<Elem>& A,
                       int i, int j, int m) {
   if (i < j) {
      int pidx = select_pivot(A, i, j);
      int k;
      partition(A, pidx, i, j, k);
      // A[i..k-1] < A[k] < A[k+1..j]
      partial_quicksort(A, i, k - 1, m);
      if (k < m - 1) // 'A' starts at index 0
        partial_quicksort(A, k + 1, j, m);
   }
```

Algorithm 1: Partial quicksort

3 The average number of comparisons

Let $P_{n,m}$ denote the average number of (key) comparisons made by partial quicksort to sort the m smallest elements out of n. Let $\pi_{n,k}$ denote the probability that the chosen pivot is the kth element among the n given elements. We assume, as it is usual in the analysis of comparison-based sorting algorithms, that any permutation of the given distinct n elements is equally likely.

[1] We assume for simplicity that all the elements in the array are distinct. Only minor modifications are necessary to cope with duplicate elements.

Then

$$P_{n,m} = n - 1 + \sum_{k=1}^{m} \pi_{n,k} \left(P_{k-1,k-1} + P_{n-k,m-k}\right) + \sum_{k=m+1}^{n} \pi_{n,k} P_{k-1,m}, \qquad \text{if } n > 0, \tag{3.1}$$

and $P_{0,m} = 0$, otherwise.

As we have already mentioned, partial quicksort behaves exactly as quicksort when $m = n$, so that $P_{n,n} = q_n = 2(n+1)H_n - 4n$, where $H_n = \sum_{1 \leq k \leq n} 1/k = \ln n + \mathcal{O}(1)$ denotes the nth harmonic number [9, 10, 14]. Let

$$t_{n,m} = n - 1 + \sum_{k=0}^{m-1} \pi_{n,k+1}\, q_k.$$

Hence, we get the recurrence

$$P_{n,m} = t_{n,m} + \sum_{k=1}^{m} \pi_{n,k} P_{n-k,m-k} + \sum_{k=m+1}^{n} \pi_{n,k} P_{k-1,m}, \tag{3.2}$$

which, except for the *toll function* $t_{n,m}$, has the same form as the recurrence for the average number of comparisons made by quickselect (see [7, 10]). The recurrence above holds for whatever pivot selection scheme we use: for instance, median-of-three [6, 18], median-of-$(2t+1)$ [13], proportion-from-s [12], pseudomedian-of-9 (also called *ninther*) [1],.... However, as we have already pointed out, we will only consider the basic variant, hence, for the rest of this paper we take $\pi_{n,k} = 1/n$, for $1 \leq k \leq n$.

The techniques that we use to solve recurrence (3.2) (i.e., to find a closed form for $P_{n,m}$) are fairly standard and rely heavily on the use of generating function as the main tool. Sedgewick and Flajolet's book [17] and Knuth's *The Art of Computer Programming* [8, 9] are excellent starting points which describe in great detail these techniques.

First, we introduce the bivariate generating functions (BGFs) associated to the quantities $P_{n,m}$ and $t_{n,m}$:

$$P(z,u) = \sum_{n \geq 0} \sum_{1 \leq m \leq n} P_{n,m} z^n u^m,$$
$$T(z,u) = \sum_{n \geq 0} \sum_{1 \leq m \leq n} t_{n,m} z^n u^m.$$

We can then translate (3.2) into a functional relation over the BGFs, namely,

$$\frac{\partial P(z,u)}{\partial z} = \frac{P(z,u)}{1-z} + \frac{u}{1-uz} P(z,u) + \frac{\partial T(z,u)}{\partial z}, \tag{3.3}$$

whose solution is

$$P(z,u) = \frac{1}{(1-z)(1-uz)} \times \left\{ \int (1-z)(1-uz) \frac{\partial T(z,u)}{\partial z}\, dz + K \right\},$$

subject to $P(0,u) = 0$, as $P_{0,m} = 0$ for any m (see for instance [11]).

Since $t_{n,m} = n - 1 + \frac{1}{n}\sum_{0\le k<m} q_k$, we can decompose $P(z,u)$ as $P(z,u) = F(z,u) + S(z,u)$, where $F(z,u)$ accounts for the selection part of the toll function $(n-1)$ and $S(z,u)$ for the sorting part of the toll function $((1/n)\cdot\sum_{0\le k<m} q_k)$. That is, with $T(z,u) = T_F(z,u) + T_S(z,u)$, we have

$$T_F(z,u) = \sum_{n\ge 0} \sum_{1\le m\le n} (n-1) z^n u^m,$$

$$T_S(z,u) = \sum_{n\ge 0} \sum_{1\le m\le n} \left(\frac{1}{n} \sum_{0\le k<m} q_k \right) z^n u^m,$$

and because of linearity,

$$F(z,u) = \frac{1}{(1-z)(1-uz)} \times \left\{ \int (1-z)(1-uz) \frac{\partial T_F(z,u)}{\partial z}\, dz + K_F \right\},$$

$$S(z,u) = \frac{1}{(1-z)(1-uz)} \times \left\{ \int (1-z)(1-uz) \frac{\partial T_S(z,u)}{\partial z}\, dz + K_S \right\}.$$

Also, from the combinatorics of the problem, we have $F(0,u) = 0$ and $S(0,u) = 0$, since $[z^0u^m]F(z,u) = F_{0,m} = 0$ and $[z^0u^m]S(z,u) = S_{0,m} = 0$. This means that $K_F = -4$ and therefore $F(z,u)$ is exactly the same BGF as the one for the average number of comparisons made by standard quickselect to find the mth element out of n elements. Namely,

$$F(z,u) = \frac{4u-2}{(1-u)(1-z)(1-uz)} \ln\frac{1}{1-z} + \frac{2u-4}{(1-u)(1-z)(1-uz)} \ln\frac{1}{1-uz} + \frac{2}{(1-u)(1-z)^2(1-uz)} + \frac{2}{(1-u)(1-z)(1-uz)^2}.$$

Extracting the coefficients $F_{n,m} = [z^n u^m]F(z,u)$, we get the well-known result [7]

$$F_{n,m} = 2\big(n+3+(n+1)H_n - (m+2)H_m - (n+3-m)H_{n+1-m}\big), \quad \text{if } 1\le m\le n, \tag{3.4}$$

and $F_{n,m} = 0$ otherwise.

On the other hand,

$$\begin{aligned}\frac{\partial T_S}{\partial z} &= \sum_{n>0} z^{n-1} \sum_{0\le m<n} u^{m+1} \sum_{k=0}^{m} q_k \\ &= u \sum_{m\ge 0} u^m \sum_{k=0}^{m} q_k \sum_{n\ge m} z^n = \frac{u}{1-z} \sum_{m\ge 0} (uz)^m \sum_{k=0}^{m} q_k \\ &= \frac{u}{1-z}\frac{Q(uz)}{1-uz},\end{aligned}$$

with $Q(z) = \sum_{n\ge 0} q_n z^n$. It is not difficult to show that

$$Q(z) = \frac{2}{(1-z)^2}\left(\ln\frac{1}{1-z} - z\right),$$

and thus

$$\begin{aligned}S(z,u) &= \frac{1}{(1-z)(1-uz)} \left\{ \int u\, Q(uz)\, dz + K_S \right\} \\ &= \frac{2}{(1-uz)^2(1-z)} \ln\frac{1}{1-uz} + \frac{2}{(1-z)(1-uz)} \ln\frac{1}{1-uz} - \frac{4}{(1-uz)^2(1-z)} + \frac{4}{(1-uz)(1-z)},\end{aligned}$$

since $S(0,u) = 0$ and hence $K_S = 4$.

Extracting the coefficients $S_{n,m}$ from $S(z,u)$ is straightforward from

$$[z^n u^m] \frac{\ln\frac{1}{1-uz}}{(1-uz)^2(1-z)} = \begin{cases} (m+1)H_m - m, & \text{if } 1\le m\le n, \\ 0, & \text{otherwise.}\end{cases}$$

$$[z^n u^m] \frac{\ln\frac{1}{1-uz}}{(1-uz)(1-z)} = \begin{cases} H_m, & \text{if } 1\le m\le n, \\ 0, & \text{otherwise.}\end{cases}$$

We have then

$$S_{n,m} = [z^n u^m]S(z,u) = 2(m+1)H_m - 6m + 2H_m, \tag{3.5}$$

whenever $1\le m\le n$, and $S_{n,m} = 0$ otherwise. Finally, adding (3.4) and (3.5) we get

$$P_{n,m} = 2n + 2(n+1)H_n - 2(n+3-m)H_{n+1-m} - 6m + 6, \tag{3.6}$$

if $1 \leq m \leq n$, and $P_{m,n} = 0$ otherwise. As a further check, the reader can easily verify that $P_{n,n} = q_n$.

Now we can compare the average number of comparisons of partial quicksort $P_{n,m}$ with that of quickselsort, that is, $F_{n,m} + q_{m-1}$. And it turns out that partial quicksort makes

$$2m - 4H_m + 2$$

comparisons less than its alternative; this is probably irrelevant for small m, but significative enough when m is large, say $m = \Omega(\sqrt{n})$.

4 The average number of exchanges

In order to investigate the average number of exchanges and other quantities of interest, we generalize our analysis from the previous section to cope with toll functions of the form

$$t_{n,m} = an + b + \sum_{k=0}^{m-1} \pi_{n,k} q'_k,$$

where a and b are arbitrary constants and q'_n is the solution of

$$\begin{aligned} q'_n &= an + b + 2 \sum_{0 \leq k < n} \pi_{n,k} q'_k, \\ q'_0 &= 0. \end{aligned}$$

Observe that we assume now that the cost of the non-recursive part (selecting the pivot, partitioning, etc.) is given by $an+b$; hence the average cost of "quicksorting" the left subarray when $k \leq m$ must be calculated using the same toll function.

If we only wish to compare partial quicksort with quickselsort, it is now quite clear that we have only to compute the coefficients of $S(z,u)$, but now we have to use the generating function of the sequence $\{q'_n\}_{n \geq 0}$:

$$Q(z) = \frac{1}{(1-z)^2} \left(2a \ln \frac{1}{1-z} + (b-a)z \right).$$

Hence,

$$\begin{aligned} S(z,u) &= \frac{1}{(1-z)(1-uz)} \left\{ \int u\, Q(uz)\, dz + K_S \right\} \\ &= \frac{2a}{(1-uz)^2(1-z)} \ln \frac{1}{1-uz} \\ &\quad + \frac{a-b}{(1-z)(1-uz)} \ln \frac{1}{1-uz} \\ &\quad - \frac{b-3a}{(1-uz)^2(1-z)} + \frac{3a-b}{(1-uz)(1-z)}, \end{aligned}$$

since $K_S = 3a - b$.

The quantity of interest is then

$$\begin{aligned} [z^n u^m] S(z,u) &= 2amH_m + (3a-b)H_m \\ &\quad + (b-5a)m, \quad \text{if } 1 \leq m \leq n, \end{aligned}$$

and if we compare q'_{m-1} with the result above, we get that partial quicksort makes

$$2am + (b-3a)H_m + a - b \tag{4.7}$$

operations less than quickselsort. For instance, for the partitioning method given in [14], the average number of exchanges made during a single partitioning stage is $n/6 - 1/3$. Substituting these values ($a = 1/6$, $b = -1/3$) in (4.7), we find that we save $\frac{1}{3}m - \frac{5}{6}H_m + \frac{1}{2}$ exchanges by using partial quicksort.

5 Conclusions

In this short research note we have presented a preliminary analysis of partial quicksort which exhibits its competitive advantage over common, straightforward alternatives for the partial sort problem. It also shows that partial quicksort can be a rich source of new and appealing mathematical problems. Sections 3 and 4 are witness of the power of the standard and by now classic tools of the analysis of algorithms.

Partial quicksort can be subject to standard optimization techniques like sampling, recursion removal, recursion cutoff on small subfiles, and so on. Also, the idea behind partial quicksort applies to similar algorithms like radix sort [9] or quicksort for strings [2].

In the author's opinion, a challenging open problem on partial quicksort is to find a combinatorial, intuitive explanation for the difference between partial quicksort and quickselsort. It is not self-evident that partial quicksort does significantly ($\Theta(m)$) less work on the average than quickselsort, even though it is more or less clear that partial quicksort would not do more work.

Partial quicksort sorts the m elements incrementally, in chunks, while it looks for the mth element, whereas quickselsort makes the initial recursive call to quicksort on the chunk of $m-1$ elements smaller than the mth. That means that the pivots used to find the mth element and that are to its left ($\Theta(\log m)$ on average) will be again compared, exchanged, etc. by the quicksort call while this is not the case with partial quicksort. Also, it seems that by "breaking" the sorting of the $m-1$ smallest elements in the way partial quicksort does, it makes bad partitions at early stages more unlikely and thus reduces somewhat the average complexity.

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