Comparison of Two CDS Algorithms on Random Unit Ball Graphs

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Abstract

This paper compares asymptotic "average case" performance of two closely related algorithms for finding small connected dominating sets. The stochastic model is that instances are random unit ball graphs formed from n random points in an $\ell_n \times \ell_n \times \ell_n$ cube. The first algorithm, widely known as "Rule 1", is proved to be ineffective asymptotically: if $\ell_n = O(\sqrt[3]{\frac{n}{\log n}})$, then with asymptotic probability one Rule 1 selects a dominating set that consists of all but $o(n)$ nodes in the network. In contrast, the expected size of Dai Li and Wu's Rule 4 ([30],[9]) dominating set is $\Theta(\ell_n^3)$. This latter performance is optimal insofar as the *minimum* connected dominating set also has $\Theta(\ell_n^3)$ vertices 'on average'. These conclusions are three dimensional analogues of the two dimensional results in [17] and [18].

1 Introduction

Unit ball graphs can be used as a crude model for the links between nodes in a wireless network. A *unit ball graph* $G = (V, E)$ is an undirected graph whose vertex set V is a set of points in $\Re^3$. Given V, the edge set E of a unit ball graph is determined as follows: an edge $e = \{u, v\} \in E$ joins vertices $u, v \in V$ if and only if the distance between u and v in $\Re^3$ is less than or equal to one.

In any graph $G = (V, E)$, a *dominating set* is a subset D of the vertices such that *every* $v \in V$ is either an element of D, or the neighbor of a vertex in D. A *connected dominating set* is a dominating set with the additional property that it induces a connected subgraph of G. It is clearly not possible for G to have a connected dominating set if G itself is not connected. We write "CDS" for a dominating set D such that the subgraph induced by D has the same number of components as G has. In this paper we use a random unit ball graph model, $\mathcal{G}_n$, which is connected with asymptotic probability one. Hence, with probability $1 - o(1)$, any CDS for $\mathcal{G}_n$ is also connected.

Recently there has been considerable interest in algorithms that select a small CDS [8],[27],[30],[28]. Wu Li and Dai proposed Rule k, a family of localized approximation algorithms. For each k, Rule k finds a CDS in the input graph, and a measure of the algorithms performance is the size of the dominating set that it finds. In previous work, we estimated the expected size of the Rule k dominating set, for each $k \geq 1$, in a random unit disk graph model. The purpose of this paper is to extend these results to three dimensions. The algorithm itself is not new; it is just a simplified version of Dai Li and Wu's algorithm. What is new in our work is the mathematical analysis of the algorithms' asymptotic performance in a random unit ball graph model.

For each $k \geq 1$, the CDS that is selected by the Rule k algorithm will be denoted $\mathcal{D}_k$, and its elements will be called "gateway nodes". For any vertex v, let $N(v)$ denote the set of vertices consisting of v and all the vertices that are adjacent to v. If the the vertex set is $V = \{x_1, x_2, \ldots, x_n\}$, then the Rule k dominating set $\mathcal{D}_k$ consists of all vertices $x_i \in V$ that are not excluded under the following version of Rule k:

Rule k: *Vertex x_i is excluded from $\mathcal{D}_k$ iff $N(x_i)$ contains at least one set of k vertices $x_{i_1}, x_{i_2}, \ldots x_{i_k}$ such that*

- $i_1 > i_2 > \cdots > i_k > i$, *and*
- *the subgraph induced by $\{x_{i_1}, x_{i_2}, \cdots, x_{i_k}\}$ is connected, and*
- $N(x_i) \subseteq \bigcup_{t=1}^{k} N(x_{i_t})$.

Next we define the random unit ball graphs that the algorithm acts on. Let ℓ_n be an increasing sequence of positive real numbers. Independently select random vertices $X_1, X_2, \ldots, X_n$ from a uniform distribution on an $\ell_n \times \ell_n \times \ell_n$ cube $\mathcal{Q}_n$ in $\Re^3$. Let $\mathcal{G}_n$ be the random unit ball graph that is formed from these vertices by putting an edge between two vertices iff the Euclidean distance between the two vertices is less than or equal to one. This construction determines a probability measure $\mathbf{P}_n$ on unit ball graphs formed from n vertices in $\mathcal{Q}_n$. Restrictions on the growth rates of the numbers ℓ_n will be included in the statements of theorems. To put these growth rates in context, we remark that the threshold

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for connectivity is $\ell_n = \Theta(\sqrt[3]{n/\log n})$; if ℓ_n grows faster than this, then the network will be disconnected with high probability. This topic is treated more extensively in Penrose[26].

For the remainder of this paper we adopt the following notation. For any points p and q in $\Re^3$, let $d(p,q)$ denote the ordinary Euclidean distance between p and q in $\Re^3$. For any point $p \in \Re^3$ and any compact subset $\mathcal{C} \subset \Re^3$, we define $d(p,\mathcal{C}) = \inf\{d(p,z) : z \in \mathcal{C}\}$. Finally, for any $r > 0$, and any $p \in \Re^3$, let $B_r(p) = \{w \in \Re^3 | d(p,q) \le r\}$ be a closed ball of radius r, centered at p. Our notation (such as $\mathcal{A}_m$ for an event) is not global, but rather is defined differently in different sections of the paper.

2 Local Coverage by One Point

In this section we define another random graph $\mathcal{H}_m$, and use it to prove the crucial Lemma 2.2 below. Fix $z \in \Re^3$; without loss of generality $z = (0,0,0)$. Given $m > 0$ and let $V(\mathcal{H}_m) = \{P_1, P_2, \ldots, P_m\}$ be a set of m points sampled independently and uniformly from $B_1(z)$. Form a random graph $\mathcal{H}_m$ with vertex set $V(\mathcal{H}_m)$ such that any two points in $V(\mathcal{H}_m)$ are adjacent iff the Euclidean distance between them is less than or equal to one. Let $\mathbf{L} = \mathbf{L}_{m,z}$ be the corresponding probability measure on unit ball graphs. Note that $\mathcal{H}_m$ is not the same as $\mathcal{G}_m$ since the vertices of $\mathcal{H}_m$ lie in a small ball rather than in a large cube.

From the construction of $\mathcal{H}_m$ it is clear that it is *possible* for a single point $P_i \in V(\mathcal{H}_m)$ to have degree $m-1$ in $\mathcal{H}_m$ and therefore be a one-point dominating set for $\mathcal{H}_m$. For example, if some P_i happens to coincide with z, the center of the unit ball, then P_i is adjacent to P_j for all $j \ne i$. However this is very rare: we prove that, with asymptotic probability one (as $m \to \infty$), there is no one point dominating set for $\mathcal{H}_m$.

The following fact will be needed in the proof. It is well-known, but we do not know suitable reference. In any case, it is easily verified using calculus:

Lemma 2.1. *If p,q are points in $\Re^3$ such that $d(p,q) = s \le 2$, then*

$$\text{VOLUME}\big(B_1(p) \cap B_1(q)\big) = \frac{4\pi}{3} - \frac{4\pi}{3}\left(\frac{3s}{4} - \frac{s^3}{16}\right).$$

Now let W_m be the size of the smallest dominating set in $\mathcal{H}_m$, i.e. the smallest set $\mathcal{D} \subseteq V(\mathcal{H}_m)$ of vertices with the property that all m vertices are within distance 1 of at least one of the points in $\mathcal{D}$.

Lemma 2.2. $\mathbf{L}(W_m = 1) = O(\frac{\log^3 m}{m^2})$.

Proof. Let $\gamma_m := \mathbf{L}(V(\mathcal{H}_m) \subseteq B_1(P_m))$ be the probability that all the $m-1$ points $P_1, P_2, \ldots, P_{m-1}$ are contained in $B_1(P_m)$. By Boole's inequality,

$$\mathbf{L}(W_m = 1) \le m\gamma_m. \tag{2.1}$$

Let R be the distance from P_m to z, and let $Vol(R)$ be the volume of $B_1(z) \bigcap B_1(P_m)$. Note that R is a random variable with density $f_R(s) = 4\pi s^2$. It follows that

$$\gamma_m = \int_0^1 4\pi s^2 \left(\frac{Vol(s)}{(4\pi/3)}\right)^{m-1} ds. \tag{2.2}$$

Let $\xi = \xi_m = \frac{48 \log m}{11 m}$. We break the integral into two pieces:

$$\gamma_m = I_1 + I_2, \tag{2.3}$$

where $I_1 = \int_0^\xi$ and $I_2 = \int_\xi^1$. To estimate I_1, we use the crude estimate $\frac{Vol(\rho)}{(4\pi/3)} \le 1$:

$$I_1 = \int_0^\xi 4\pi s^2 \left(\frac{Vol(s)}{(4\pi/3)}\right)^{m-1} ds \le \int_0^\xi 4\pi s^2 ds = O(\xi^3).$$

For $s \in [\xi, 1]$, we have $s^3 \le s$, and consequently $\frac{3s}{4} - \frac{s^3}{16} \ge \frac{11s}{16}$. This and Lemma 2.1 together imply that $\frac{Vol(s)}{4\pi/3} \le (1 - \frac{11s}{16}) \le (1 - \frac{11\xi}{16}) = (1 - \frac{3\log m}{m})$. Hence

$$I_2 \le \int_\xi^1 4\pi s^2 (1 - \frac{3\log m}{m})^{m-1} ds = O(\frac{1}{m^3}). \tag{2.4}$$

Putting our estimates for I_1 and I_2 in to (2.3), we get

$$\mathbf{L}(W_m = 1) \le m\gamma_m = O(\frac{\log^3 m}{m^2}). \tag{2.5}$$

3 Analysis of Rule 1

In this section we investigate the average size of $\mathcal{D}_1(\mathcal{G}_n)$, the CDS constructed by applying Rule 1 to the random graph $\mathcal{G}_n$. Recall that initially *all* vertices in $\mathcal{G}_n$ start as 'gateway' nodes. According to Rule 1, a node $x_i \in V(\mathcal{G}_n)$ becomes a non-gateway node only if it is adjacent to some $x_j \in V(\mathcal{G}_n)$ such that $j > i$ and $N(x_i) \subseteq N(x_j)$. We prove that, in the random graph $\mathcal{G}_n$, with high probability, most nodes do not have such a neighbor, i.e. most nodes remain as gateways and Rule 1 does not significantly reduce the size of the CDS.

Let U_1 be the number of vertices in $\mathcal{G}_n$ that become non-gateways under Rule 1, i.e. $U_1 = n - |\mathcal{D}_1(\mathcal{G}_n)|$. Then we have

THEOREM 3.1. *If* $\lim_{n\to\infty} \ell_n = \infty$, *but* $\ell_n = O(\sqrt[3]{n/\log n})$, *then* $E(U_1) = o(n)$ *as* $n \to \infty$.

Proof. Define the indicator variable I_i so that $I_i = 1$ iff $X_i \in V(\mathcal{G}_n)$ is adjacent to some node $X_j \neq X_i$ such that $N(X_i) \subseteq N(X_j)$. We note that the event $[I_i = 1]$ is a *necessary* but *not sufficient* condition for node i to become a non-gateway node, thus $U_1 \leq \sum_{i=1}^{n} I_i$. The advantage of this bound is that the $I_1, I_2, \ldots, I_n$ are identically distributed and

$$E(U_1) \leq n\mathbf{P}_n(I_1 = 1). \tag{3.6}$$

In this section, let $\mathcal{A}_1$ be the event that that $B_1(X_1) \subseteq \mathcal{Q}_n$, i.e. that vertex X_1 is not one of the exceptional vertices near the border of the region $\mathcal{Q}_n$. Let ρ_1 be the degree of vertex X_1, i.e. number of nodes of $\mathcal{G}_n$ in $B_1(X_1)$ other than vertex X_1 itself, and define

$$\mu = E(\rho_1|\mathcal{A}_1) = \frac{(n-1)(4\pi/3)}{\ell_n^3}. \tag{3.7}$$

Also let $\mathcal{B}_1$ be the event that $\rho_1 > \mu/2$, and let $\mathcal{C}_1 = \mathcal{A}_1 \bigcap \mathcal{B}_1$. Then, $\mathbf{P}_n(I_1 = 1) =$

$$\mathbf{P}_n(I_1 = 1|\mathcal{C}_1)\mathbf{P}_n(\mathcal{C}_1) + \mathbf{P}_n(I_1 = 1|\mathcal{C}_1^c)\mathbf{P}_n(\mathcal{C}_1^c)$$

$$\leq \mathbf{P}_n(I_1 = 1|\mathcal{C}_1) + \mathbf{P}_n(\mathcal{C}_1^c). \tag{3.8}$$

The plan is to bound the right side of (3.8) and then use the bound in (3.6). Since $\mathcal{C}_1 = \mathcal{A}_1 \cap \mathcal{B}_1$, we have

$$\mathbf{P}_n(\mathcal{C}_1^c) = \mathbf{P}_n(\mathcal{A}_1^c) + \mathbf{P}_n(\mathcal{A}_1)\mathbf{P}_n(\mathcal{B}_1^c|\mathcal{A}_1) = \tag{3.9}$$

$$= (1 - \left(\frac{\ell_n - 2}{\ell_n}\right)^3) + \left(\frac{\ell_n - 2}{\ell_n}\right)^3 \mathbf{P}_n\left(\rho_1 < \frac{\mu}{2}|\mathcal{A}_1\right).$$

By Chernoff's inequality,

$$\mathbf{P}_n\left(\rho_1 < \frac{\mu}{2}|\mathcal{A}_1\right) < e^{-\mu/8}. \tag{3.10}$$

Therefore

$$\mathbf{P}_n(\mathcal{C}_1^c) = O\left(\frac{1}{\ell_n}\right) + O(e^{-n/2\ell_n^3}). \tag{3.11}$$

For the remaining term on the right side of (3.8), we write

$$\mathbf{P}_n(I_1 = 1|\mathcal{C}_1) = \tag{3.12}$$

$$\sum_{m>\mu/2} \mathbf{P}_n(I_1 = 1|\rho_1 = m, \mathcal{A}_1)\mathbf{P}_n(\rho_1 = m, \mathcal{A}_1|\mathcal{A}_1 \cap \mathcal{B}_1).$$

By Lemma 2.2 there is a constant $C > 0$ such that, for all $m > \mu/2$,

$$\mathbf{P}_n(I_1 = 1|\rho_1 = m, \mathcal{A}_1) < \frac{C\log^3 m}{m^2} \leq \frac{C\log^3 \mu}{\mu^2}.$$

Putting this bound into (3.12) on m, and summing on m, we get S

$$\mathbf{P}_n(I_1 = 1|\mathcal{C}_1) = O\left(\frac{\log^3 \mu}{\mu^2}\right). \tag{3.13}$$

Recall that $\mu = \Theta(n/\ell_n^3)$. Putting (3.13) back into (3.8), we get

$$\mathbf{P}_n(I_1 = 1) = O\left(\frac{\log^3 \mu}{\mu^2}\right) + O\left(\frac{1}{\ell_n}\right) + O(e^{-n/2\ell_n^3}),$$

and therefore

$$E(U_1) \leq n\mathbf{P}_n(I_1 = 1) = o(n).$$

4 Local Coverage by Four Vertices

The next lemma is a geometric result which is needed for the proof of Theorem 4.1. Let z be any point in $\mathcal{Q}_n$, and let $B' = B_1(z) \cap Q_n$ be the set of points in the cube $\mathcal{Q}_n$ whose distance from z is less than or equal to one.

LEMMA 4.1. *There exist points* $z_1, z_2, z_3, z_4 \in B'$ *such that*

- $B' \subseteq \bigcup_{i=1}^{4} B_{.98}(z_i)$, *and*
- *for any* $1 \leq i \neq j \leq 4$, $d(z_i, z_j) \leq 2\sqrt{2}/3$.

Proof. We begin by considering the case where $B_1(z) \subseteq \mathcal{Q}_n$, i.e. z is not near the boundary of the cube $\mathcal{Q}_n$. We may, without loss of generality, choose the coordinate system such that $z = (0, 0, 0)$ and such that each axis is parallel to one on the edges of the cube $\mathcal{Q}_n$. Define four points $z_1 = (\frac{1}{3}, \frac{-1}{3}, \frac{-1}{3})$, $z_2 = (\frac{-1}{3}, \frac{1}{3}, \frac{-1}{3})$, $z_3 = (\frac{-1}{3}, \frac{-1}{3}, \frac{1}{3})$, and $z_4 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. These are non-adjacent vertices of a cube that is centered at the origin and has edges of length $\frac{2}{3}$. Hence they are the vertices of a regular tetrahedron. (We thank Ron Perline for suggesting that we try a tetrahedron.) For each $1 \leq i \leq 4$, the half-line that begins at z_i and passes through the origin meets the surface of the unit sphere $\partial B_1((0, 0, 0))$ at a unique point ζ_i. For example, $\zeta_4 = (\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}})$. It follows from straightforward (though somewhat tedious) geometrical calculations that the points $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ are the four points in $B_1(z)$ that are 'most distant' from the set $\{z_1, z_2, z_3, z_4\}$: for any any $x \in B_1(z)$, and any i,

$$d(x, \{z_1, z_2, z_3, z_4\}) \leq d(\zeta_i, \{z_1, z_2, z_3, z_4\})$$

$$= \sqrt{\frac{12 - 2\sqrt{3}}{9}} < .98.$$

So the result holds in this case.

Next we consider the case where $z \in \mathcal{Q}_n$ but $B_1(z)$ is not contained in $\mathcal{Q}_n$. Again, choose the coordinate system and the points z_1, z_2, z_3, z_4 as above. Since $B_1(z) \cap \mathcal{Q}_n \equiv B' \neq B_1(z)$, one or more of the points z_1, z_2, z_3, z_4 may not lie in $\mathcal{Q}_n$. In particular, if $z_k \notin \mathcal{Q}_n$, then there is a (unique) $z'_k \in \mathcal{Q}_n$ such that $d(z_k, z'_k) = d(z_k, \mathcal{Q}_n)$. We replace z_k by z'_k and observe that every point of B' is closer to z'_k than it is to the original point z_k, i.e. if $x \in B_{0.98}(z_k) \cap \mathcal{Q}_n$ then $x \in B_{0.98}(z'_k) \cap \mathcal{Q}_n$. After replacing all z_k such that $z_k \notin \mathcal{Q}_n$ by the corresponding z'_k we obtain four points that satisfy the conditions of the lemma.

Now fix $z \in \mathcal{Q}_n$ and let $P_1, P_2, \ldots, P_m$ be m points sampled uniformly and independently from $B' = B_1(z) \cap \mathcal{Q}_n$. In this section, let $\mathcal{A}_m$ be the event that there exist $1 \leq i_1 < i_2 < i_3 < i_4 \leq m$ such that

- $B' \subseteq \cup_{k=1}^4 B_1(P_{i_k})$, and
- the unit ball graph with vertices $P_{i_1}, P_{i_2} P_{i_3}, P_{i_4}$ is connected.

Our goal is to prove that $\mathcal{A}_m$ occurs with high probability:

Theorem 4.1. *There is a constant $0 < \alpha < 1$ (which is independent of n and of the location of z in $\mathcal{Q}_n$) such that, for all positive integers m, $\Pr(\mathcal{A}_m) > 1 - 4\alpha^m$.*

Proof. Choose points z_1, z_2, z_3, z_4 as in the proof of Lemma 4.1 and let $r = 1 - \sqrt{\frac{12-2\sqrt{3}}{9}} = .026\ldots$. For $1 \leq k \leq 4$, let $\mathcal{E}_k$ be the event that $B_r(z_k) \cap \mathcal{Q}_n$ does not contain *any* of the m points $P_1, P_2, ..., P_m$. Then

$$\text{(4.14)}\ \Pr(\mathcal{E}_k) = \left(1 - \frac{\text{Volume}(B_r(z_k) \cap \mathcal{Q}_n)}{\text{Volume}(B')}\right)^m.$$

Since $\text{Volume}(B_r(z_k) \cap \mathcal{Q}_n) \geq \frac{1}{8}\text{Volume}(B_r(z_k)) = \frac{\pi r^3}{6}$ and $\text{Volume}(B') \leq \frac{4\pi}{3}$, Boole's inequality yields

$$\Pr\left(\bigcup_{i=1}^4 \mathcal{E}_i\right) \leq 4\alpha^m. \tag{4.15}$$

where $\alpha = (1 - \frac{r^3}{8}) < 1$.

If the event $\bigcup_{k=1}^4 \mathcal{E}_k$ does *not* occur, then we can choose i_1, i_2, i_3, i_4 such that, for $1 \leq k \leq 4$, we have $P_{i_k} \in B_r(z_k)$. Let x be an arbitrary point in $B' = B_1(z) \cap \mathcal{Q}_n$, and let z_k be the point closest to x of the four points z_1, z_2, z_3, z_4. Then by our choice of the points $z_1, z_2, z_3, z_4, P_{i_1}, P_{i_2}, P_{i_3}, P_{i_4}$, and of r, we have

$$d(x, P_{i_k}) \leq d(x, z_k) + d(z_k, P_{i_k}) \leq (1 - r) + r = 1.$$

We also have for $1 \leq k < j \leq 4$, $d(P_{i_k}, P_{i_t}) \leq$

$$d(P_{i_k}, z_k) + d(z_k, z_j) + d(z_j, P_{i_j}) \leq 2r + \frac{2\sqrt{2}}{3} < 1.$$

Hence $(\cup_{k=1}^4 \mathcal{E}_k)^c \subseteq \mathcal{A}_m$ and the result follows from inequality (4.15).

5 Analysis of Rule 4.

Now that Theorem 4.1 is proved, the remaining analysis of Rule 4 is quite similar to the 2D case [17]. Let $U_4 = \sum_{i=1}^n I_i$, where (in this section) the indicator variable $I_i = 1$ iff node i is excluded from $\mathcal{D}_4(\mathcal{G}_n)$ under Rule 4. Thus Rule 4 selects a dominating set having $|\mathcal{D}_4| = n - U_4$ vertices, and it is desirable for U_4 to be large. Our goal in this section is to prove that $E(U_4) \geq n - O(\ell_n^3)$.

Let $X_1, X_2, \ldots, X_n$ be uniform random points in $\mathcal{Q}_n$, namely the locations of vertices $1, 2, \ldots, n$. Let $\hat{\rho}_i$ be the number of neighbors of vertex i having a larger ID, i.e. the number of $j > i$ such that $d(X_i, X_j) \leq 1$.

Lemma 5.1. *For all i, $\mathbf{P}_n(\hat{\rho}_i < \frac{(n-i)\pi}{12\ell_n^3}) \leq \exp(\frac{-(n-i)\pi}{48\ell_n^3})$*

Proof. Let $|B_1(X_i)|$ be the volume of the set of points in $\mathcal{Q}_n$ whose distance from X_i is one or less. Thus $|B_1(X_i)| = \frac{4\pi}{3}$ unless X_i happens to fall near the boundary of $\mathcal{Q}_n$, and in all cases $|B_1(X_i)| \geq \frac{\pi}{6}$. Given $|B_1(X_i)|$, the variable $\hat{\rho}_i$ has a Binomial$\big(n - i, \frac{|B_1(X_i)|}{\ell_n^3}\big)$ distribution. Therefore Chernoff's bound on the lower tail distribution gives

$$\mathbf{P}_n\left(\hat{\rho}_i < \frac{(n-i)\pi}{12\ell_n^3}\,\middle|\, |B_1(X_i)|\right) \leq$$

$$\mathbf{P}_n\left(\hat{\rho}_i < \frac{(n-i)|B_1(X_i)|}{2\ell_n^3}\,\middle|\, |B_1(X_i)|\right)$$

$$\leq \exp\left(\frac{-(n-i)|B_1(X_i)|}{8\ell_n^3}\right)$$

$$\leq \exp\left(\frac{-(n-i)\pi}{48\ell_n^3}\right). \tag{5.16}$$

Since the bound on the right hand side of (5.16) is uniform over all possible values of $|B_1(X_i)|$, the result follows.

Theorem 5.1. $E(U_4) \geq n - O(\ell_n^3)$.

Proof. For $1 \le i \le n$, let $\mathcal{B}_i$ be the event that $\hat{\rho}_i \ge \frac{(n-i)\pi}{12\ell_n^3}$. By Lemma 5.1,

$$\mathbf{P}_n(I_i = 1) \ge \mathbf{P}_n(I_i = 1|\mathcal{B}_i)\mathbf{P}_n(\mathcal{B}_i) \ge$$

$$\mathbf{P}_n(I_i = 1|\mathcal{B}_i)\Big(1 - \exp\left(\frac{-(n-i)\pi}{48\ell_n^3})\right)\Big). \tag{5.17}$$

Observe that $\mathbf{P}_n(I_i = 1|\mathcal{B}_i) =$

$$\sum_{v \ge \frac{(n-i)\pi}{12\ell_n^3}} \mathbf{P}_n(I_i = 1|\hat{\rho}_i = v)\mathbf{P}_n(\hat{\rho}_i = v|\mathcal{B}_i). \tag{5.18}$$

To estimate this, observe that $\mathbf{P}_n(I_i = 1|\hat{\rho}_i = v) =$

$$\int_{\mathcal{Q}_n} \mathbf{P}_n(I_i = 1|\hat{\rho}_i = v, X_i = x) f_{X_i}(x|\hat{\rho}_i = v)dx \tag{5.19}$$

where $f_{X_i}(x|\hat{\rho}_i = v)$ is the conditional density of X_i on the cube $\mathcal{Q}_n$ *given* that $\hat{\rho}_i = v$. For $v \ge (n-i)\pi/12\ell_n^3$, Theorem 4.1 yields

$$\mathbf{P}_n(I_i = 1|\hat{\rho}_i = v, X_i = x) \ge \tag{5.20}$$

$$1 - 4\alpha^v \ge 1 - 4\alpha^{(n-i)\pi/12\ell_n^3}.$$

Putting this back into (5.19) and then (5.18), we get

$$\mathbf{P}_n(I_i = 1|\mathcal{B}_i) \ge 1 - 4\alpha^{(n-i)\pi/12\ell_n^2} \tag{5.21}$$

and therefore

$$\mathbf{P}_n(I_i = 1) \ge \mathbf{P}_n(I_i = 1|\mathcal{B}_i)\mathbf{P}_n(\mathcal{B}_i) \ge$$

$$(1 - 4\alpha^{(n-i)\pi/12\ell_n^2})\Big(1 - \exp(-\frac{(n-i)\pi}{48\ell_n^3})\Big)$$

$$\ge 1 - 4\alpha^{(n-i)\pi/12\ell_n^3} - \exp(-\frac{(n-i)\pi}{48\ell_n^3})$$

$$\ge 1 - 5\alpha^{(n-i)\pi/12\ell_n^3}. \tag{5.22}$$

Finally, if we let $\lambda_n = n - \ell_n^3$, then it follows from the inequality (5.22) that

$$E(U_4) \ge \sum_{i=1}^{\lambda_n} \mathbf{P}_n(I_i = 1) \ge$$

$$\sum_{i=1}^{\lambda_n}\Big(1 - 5\alpha^{(n-i)\pi/12\ell_n^3}\Big)$$

$$= \lambda_n - 5\sum_{j=\ell_n^3}^{n}\left(\alpha^{\pi/12\ell_n^3}\right)^j$$

$$= n - \ell_n^3 + O(\ell_n^3).$$

Corollary 5.1. $E(|\mathcal{D}_4|) = O(\ell_n^3)$.

6 Lower Bound

If a vertex v has higher ID than any of its neighbors, then it cannot be eliminated under Rule k. This simple observation is the basis for

Theorem 6.1. *If $\ell_n = O(\sqrt[3]{n/\log n})$, then, for all sufficiently large n, the expected size of the Rule k dominating set is greater than ℓ_n^3.*

Proof. Let $Z = \sum_{i=1}^{n} I_i$, where (in this section) $I_i = 1$ iff node i has a higher ID that all the nodes in $B_1(X_i)$. Note that $I_i = 1$ iff the nodes $X_{i+1}, X_{i+2}, \ldots, X_n$ all fall *outside* the ball $B_1(X_i)$. Therefore

$$\mathbf{P}_n(I_i = 1) = \left(1 - \frac{|B_1(X_i)|}{\ell_n^3}\right)^{n-i} \ge \left(1 - \frac{\pi}{6\ell_n^3}\right)^{n-i}.$$

Therefore

$$E(|\mathcal{D}_4|) \ge E(Z) \ge \sum_{i=1}^{n}\left(1 - \frac{\pi}{6\ell_n^3}\right)^{n-i}$$

$$= \frac{6\ell_n^3}{\pi}\left(1 - O(e^{-\pi n/6\ell_n^3})\right).$$

Combining Theorems 7 and 6.1, we get

Corollary 6.1. $E(|\mathcal{D}_4|) = \Theta(\ell_n^3)$.

7 Discussion

It is clear from the proof that the expected size of the Rule k dominating set is $\Theta(\ell_n^3)$ for any fixed $k \ge 4$: if $k > 4$, then in the proof of Theorem 4.1 we could simply include $k-4$ redundant vertices. Limited simulations by Patricia Stamets suggest that Rule 3 may be effective and Rule 2 will not. However we do not have proofs of these conjectures; it is an open problem to estimate the expected size of the Rule 2 and Rule 3 dominating sets in the three dimensional stochastic model.

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