Counting Structures in Grid Graphs, Cylinders and Tori Using Transfer Matrices: Survey and New Results (Extended Abstract)*

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Abstract

There is a very large literature devoted to counting structures, e.g., spanning trees, Hamiltonian cycles, independent sets, acyclic orientations, in the $n \times m$ grid graph $G(n,m)$. In particular the problem of counting the number of structures in *fixed height* graphs, i.e., fixing m and letting n grow, has been, for different types of structures, attacked independently by many different authors, using a *transfer matrix approach.* This approach essentially permits showing that the number of structures in $G(n,m)$ satisfies a fixed-degree constant-coefficient recurrence relation in n.

In contrast there has been surprisingly little work done on counting structures in grid-cylinders (where the left and right, or top and bottom, boundaries of the grid are wrapped around and connected to each other) or in grid-tori (where the left edge of the grid is connected to the right *and* the top edge is connected to the bottom one). The goal of this paper is to demonstrate that, with some minor modifications, the transfer matrix technique can also be easily used to count structures in fixed height grid-cylinders and tori.

1 Introduction

Grid graphs are very common and there is an extremely large literature devoted to counting structures in them. See Table 1. Let $G(n,m)$ denote the $n \times m$ grid graph. Much of the counting literature asks questions of the type *"let m and/or n go to ∞; how does the number of spanning trees (or Hamiltonian cycles, independent sets, acyclic orientations, k-colorings, etc.) grow as a function of n and/or m.* Table 1 presents a selection of these results. Many of the results in this area work by assuming that m (the "height" of the grid) is fixed and examine how the number of structures grows as $n \to \infty$; in almost all cases the technique used follows a transfer matrix formulation (or something equivalent, e.g., recursively calculating the Tutte-polynomial of the growing fixed-height grid [12]). This very natural technique was developed independently by many authors without knowing that it had been used for solving other grid counting problems. The technique permits showing that, for fixed height-m grids, the number of designated structures in $G(n,m)$ will grow as $\vec{a}A_m^n\vec{b}^t$ where A_m is some square matrix and $\vec{a},\vec{b}$ are vectors, all with non-negative integral entries. As will be explained shortly, this immediately implies that, for fixed m, the number of structures in $G(n,m)$ satisfies a fixed-order constant coefficient recurrence relation in n, something that was not a-priori obvious.

Given the large amount of prior work on grid-graphs it is surprising to note that there seems to be very little work done on counting structures in related graphs such as cylinders or tori[1]. The main goal of this paper is to show that, by adding a little extra framework, the transfer matrix method can also easily count structures in grid cylinders and tori.

We start by formally defining the graphs and the values to be counted. See Figure 1.

DEFINITION 1.1. *The $n \times m$ grid graph* Grid Graph *$G(n,m)$, has vertex set*

$$V(n,m) = \{(i,j) \,:\, 0 \le i < n,\, 0 \le j < m\}$$

and edge set

$$E_G(n,m) = \{((i,j),\,(i',j')) \,:\, |i-i'| + |j-j'| = 1\}.$$

Let

$$\begin{aligned}\texttt{Top}(n,m) &= \{((j,m-1),\,(j,0)) \,:\, 0 \le j < n\},\\ \texttt{Side}(n,m) &= \{((n-1,i),\,(0,i)) \,:\, 0 \le i < m\}.\end{aligned}$$

Fat Cylinders $FC(n,m)$, Thin Cylinders $TC(n,m)$ *and* Tori $T(n,m)$ *are graphs with the same vertex set*

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[1] One of the few exceptions is the analysis in [4] of spanning trees in what we will later define as fat-cylinders.

$V(n,m)$ *but respective edge sets:*

$$
\begin{aligned}
E_{FC}(n,m) &= E_G(n,m) \cup \texttt{Side}(n,m) \\
E_{TC}(n,m) &= E_G(n,m) \cup \texttt{Top}(n,m) \\
E_T(n,m) &= E_G(n,m) \cup \texttt{Top}(n,m) \cup \texttt{Side}(n,m)
\end{aligned}
$$

DEFINITION 1.2. *Let $\mathcal{S}$ be one of the structures described in Table 1. Let $\mathcal{G} \in \{G, FC, TC, T\}$ be a grid, fat cylinder, thin cylinder or torus. Then $\mathcal{S}_{\mathcal{G}}(n,m)$ will be the* set of all structures of type $\mathcal{S}$ in the graph $\mathcal{G}(n,m)$ *and $|\mathcal{S}_{\mathcal{G}}(n,m)|$ will be the* number of structures of type $\mathcal{S}$ in the graph $\mathcal{G}(n,m)$*, e.g., $\mathbf{ST}_{FC}(n,m)$ is the number of spanning trees in the $n \times m$ fat-cylinder.*

The goal of this paper is to point out the following "meta-theorem",

THEOREM 1.1. *Let $\mathcal{S}$ be one of the structures listed in Table 1, $\mathcal{G} \in \{G, FC, TC, T\}$, and $m \geq 1$ an integer. Define the function in n,*

$$f(\mathcal{S}, \mathcal{G}, m; n) = |\mathcal{S}_{\mathcal{G}}(n,m)|.$$

Then there exists

- *an integer k (function of $\mathcal{S}$, $\mathcal{G}$, and m)*
- *A $k \times k$* transfer matrix $A(\mathcal{S}, \mathcal{G}, m)$ *with nonnegative integer entries*
- *two $1 \times k$ vectors $\vec{a}(\mathcal{S}, \mathcal{G}, m)$, $\vec{b}(\mathcal{S}, \mathcal{G}, m)$, with nonnegative integer entries*

such that

$$f(\mathcal{S}, \mathcal{G}, m; n) = \vec{a}(\mathcal{S}, \mathcal{G}, m)\, A^n(\mathcal{S}, \mathcal{G}, m)\; \vec{b^t}(\mathcal{S}, \mathcal{G}, m) \tag{1.1}$$

As an example, if $\mathcal{S} = \mathbf{HC}$ and $\mathcal{G} = T$ (torus) then the theorem says that, for fixed m, the number of Hamiltonian Cycles in a $n \times m$ torus grows as $\vec{a}A^n\vec{b^t}$ for some fixed integral matrix A and vectors $\vec{a}, \vec{b}$. For the grid graph this general technique is well known (see all of the results referenced in Table 1) but for the other graphs this does not seem to have been commented on before.

Also note that Theorem 1.1 implies that $f(\mathcal{S}, \mathcal{G}, m; n) = |\mathcal{S}_{\mathcal{G}}(n,m)|$ satisfies a fixed-degree constant-coefficient recurrence relation in n. To see this drop the labelling and write $f(n) = \vec{a}A^n\vec{b^t}$. Let $Q(x) = \sum_{i=0}^{t} q_i x_i$ be any polynomial that annihilates A, i.e., $Q(A) = 0$ (by the Cayley-Hamiltonian theorem the characteristic polynomial of A must annihilate A so such a polynomial exists.). Then it is easy to see that $\forall n \geq t$,

$$
\begin{aligned}
\sum_{i=0}^{t} q_i f(n+i) &= \vec{a}\left(\sum_{i=0}^{t} q_i A^{n+i}\right)\vec{b^t} \\
&= \vec{a}A^n\left(\sum_{i=0}^{t} q_i A^{i}\right)\vec{b^t} \\
&= \vec{a}A^n\,\mathbf{0}\,\vec{b^t} = 0
\end{aligned}
$$

where $\mathbf{0}$ denotes the $k \times k$ zero matrix and 0 a scalar so $f(n)$ satisfies the degree-t constant coefficient recurrence relation $f(n+t) = \sum_{i=0}^{t-1} -\frac{q_i}{q_t} f(n+i)$ in n.

As an interesting side note we point out that, for all of the problems $\mathcal{S}$ listed in Table 1 (with the exception of Eulerian Orientations and Eulerian Tours which are not well defined except on the torus) our derivation will have the further property that

$$
\begin{aligned}
A(\mathcal{S}, FC, m) &= A(\mathcal{S}, G, m) \\
\vec{b}(\mathcal{S}, FC, m) &= \vec{b}(\mathcal{S}, G, m),
\end{aligned}
$$

and

$$
\begin{aligned}
A(\mathcal{S}, T, m) &= A(\mathcal{S}, TC, m) \\
\vec{b}(\mathcal{S}, T, m) &= \vec{b}(\mathcal{S}, TC, m);
\end{aligned}
$$

in particular, we will only need to build *two* transfer matrices, one shared by G and FC and a second shared by TC and T, and not four.

Before proceeding to the derivation we will need some simple observations as to how graphs grow. In particular we note that a grid/thin-cylinder of size $(n+1) \times m$ can, independently of n, be recursively built by starting with $G(n,m)/TC(n,m)$, adding the rightmost column of nodes and the appropriate set of "right-edges". Furthermore, the fat-cylinders/tori can be built from the corresponding grids/thin-cylinders by adding the edges $\texttt{Side}(n,m)$ (The fat-cylinders/tori are thus a special case of what have recently been labelled *Recursively constructible graphs* in [13].) Since these observations are at the core of our derivation we collect these facts in the following lemma (see Figure 2)

LEMMA 1.1. *Define $L(m)$ and $R(n,m)$ to be the "left-most" and "rightmost" columns of vertices in the $n \times m$ graphs we are considering*

$$
\begin{aligned}
L(m) &= \{(0,i) : i = 0, \ldots, m-1\} \\
R(n,m) &= \{(n-1,i) : i = 0, \ldots, m-1\}
\end{aligned}
$$

A grid/thin-cylinder of size $(n+1) \times m$ can be recursively built by starting with $G(n,m)/TC(n,m)$, adding the column $R(n,m)$ and the appropriate set of "right-edges". That is, defining

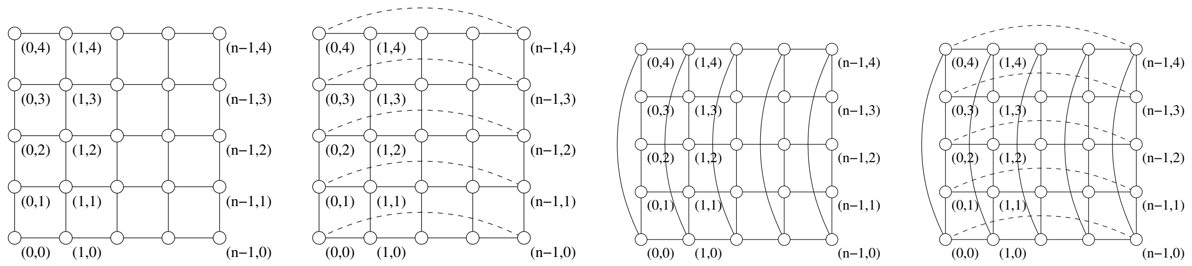
$$
\begin{aligned}
\texttt{RtG}(n,m) &= \{\, ((n-1,i),(n,i))) : 0 \leq i < m\} \\
&\quad \cup \{\, ((n,i),(n,i+1)) : 0 \leq i < m-1\} \\
\texttt{RtTC}(n,m) &= \texttt{RtG}(n,m) \cup \{((n,m-1),(n,0))\}
\end{aligned}
$$

gives

$$
\begin{aligned}
E_G(n+1,m) &= E_G(n,m) \cup \texttt{RtG}(n,m) \\
\text{and } E_{TC}(n+1,m) &= E_{TC}(n,m) \cup \texttt{RtTC}(n,m)
\end{aligned}
$$

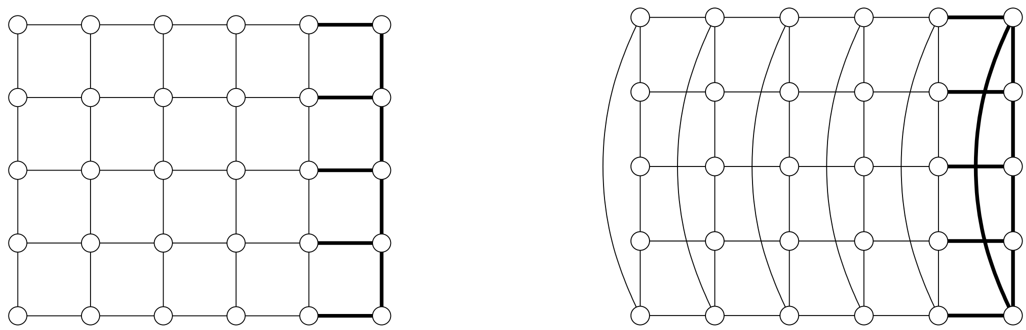
IS	*Independent Sets/2D* $(1,\infty)$ *RLL codes* $V' \in V$ s.t $\forall u, v \in V'$, $(u,v) \notin E$ Independent sets in grid graphs are in $1-1$ correspondence with 2-Dimensional $(1,\infty)$ run-length limited codes	[5] [3] [8] [14] [6] Emphasis is on deriving upper & lower bounds on $c_{\mathbf{IS}}(G)$
DM	*Dimer Matchings* A placement of 1×2 "dominos" that covers V such that a domino covers nodes u, v iff $(u,v) \in E$.	[16] [18] [7] give "closed formula" for $|\mathbf{DM}_G(n,m)|$
HC	*Hamiltonian Cycles* A simple cycle containing all of the vertices	[9] [17]
ST	*Spanning Trees* A connected acyclic subgraph containing all vertices	[12] [15] [4]
SF	*Spanning Forests* An acyclic subgraph containing all vertices	[2] [12]
EO	*Eulerian Orientations/Ice Condition* An orientation of the edges in which every vertex has indegree 2 and outdegree 2 (only defined for tori)	[10] gives closed expression for $c_{EO}(T)$
ET	*Eulerian Tours* An orientation of the edges along with a circular ordering of the edges such that the source of each edge is equal to the sink of its predecessor (only defined for tori)	this paper
CC **DCC**	*Cycle Covers/Directed Cycle Covers* **CC** is a collection of simple cycles that together contain each vertex exactly once. **DCC** is a Cycle cover along with an orientation (clockwise/counterclockwise) of each vertex	[1]
AO	*Acyclic Orientations* An orientation of the edges that contains no directed cycle.	[2] [12]
k**C**	*k Colorings* A function $f : E \to \{1, \ldots, k\}$ such that if $(u,v) \in E$ then $f(u) \neq f(v)$	[12]

Table 1: The problems addressed, short descriptions, and (a representative list of) references.
If $G = (V, E)$ is an *undirected* graph, an *edge orientation* of G transforms G into a directed graph by giving each $(u,v) \in E$ a *direction*.
DM is the only case for which a closed form in n, m is known for $|\mathcal{S}_{\mathcal{G}}(n,m)|$.
Define $c_{\mathcal{S},\mathcal{G}} = \lim_{n,m\to\infty} |\mathcal{S}_{\mathcal{G}}(n,m)|^{1/mn}$ (if the limit exists).
EO is the only case for which a closed form of $c_{\mathcal{S},\mathcal{G}}$ is known. In other problems, only upper and lower bounds on $c_{\mathcal{S},\mathcal{G}}$ have been found.



(a) $G(n,m)$ (b) $FC(n,m)$ (c) $TC(n,m)$ (d) $T(n,m)$

Figure 1: The basic graphs. Note that $E_{FC}(n,m) = E_G(n,m) \cup \texttt{Side}(n,m)$ and $E_T(n,m) = E_{TC}(n,m) \cup \texttt{Side}(n,m)$



(a) $E_G(n+1,m) = E_G(n,m) \cup \texttt{RtG}(n,m)$ (b) $E_{TC}(n+1,m) = E_{TC}(n,m) \cup \texttt{RtTC}(n,m)$

Figure 2: How grid graphs and thin-cylinders grow. The dark edges are $\texttt{RtG}(n,m)$ and $\texttt{RtTC}(n,m)$.

Finally, note that

$$\begin{aligned} E_{FC}(n,m) &= E_G(n,m) \cup \texttt{Side}(n,m) \\ \textit{and } E_T(n,m) &= E_{TC}(n,m) \cup \texttt{Side}(n,m) \end{aligned}$$

2 The General Technique

In this section we describe our technique. Due to the space limitations of this extended abstract we do not prove it for all cases. Instead we describe the general framework and, to illustrate, sketch how the structure $\mathcal{S} = \textbf{ST}$ fits into the framework.

Also, our general framework is only described for structures that are defined as *subsets* of the edges of the graphs possessing special properties. This general framework is applicable for $\mathcal{S} \in \{\textbf{HC}, \textbf{DM}, \textbf{ST}, \textbf{SF}, \textbf{CC}\}$

In the case of $\{\textbf{ET}, \textbf{AO}\}$ where the structure is an *orientation* of *all* of the edges in the graph possessing a certain property or $\{\textbf{IS}, \textbf{EO}, \textbf{kC}\}$ where the structure is a *labelling* of the *vertices* possessing a certain property, some of the notation and definitions of the general framework must be changed appropriately (but the general approach can still be used). In this extended abstract we do not give these changes but they are very straightforward.

We start by assuming that we are interested in counting the number of $\mathcal{S}$ (e.g., spanning trees) in *grid* graphs. We first show how to construct the transfer matrix $A(\mathcal{S}, G, m)$. We then show how to define $\vec{a}(\mathcal{S}, G, m)$, $\vec{b}(\mathcal{S}, G, m)$, such that

$$f(\mathcal{S}, G, m; n) = \vec{a}(\mathcal{S}, G, m)\, A^n(\mathcal{S}, G, m)\, \vec{b}^t(\mathcal{S}, G, m). \tag{2.2}$$

We then show how to solve the problem on fat-cylinders by defining $\vec{a}(\mathcal{S}, FC, m)$, for the fat-cylinders such that

$$f(\mathcal{S}, FC, m; n) = \vec{a}(\mathcal{S}, FC, m)\, A^n(\mathcal{S}, G, m)\, \vec{b^t}(\mathcal{S}, G, m) \tag{2.3}$$

(so the transfer matrix and $\vec{b}$ for grids and fat cylinders will be the same).

After this we will discuss how to modify the construction of transfer matrix $A(\mathcal{S}, G, m)$ to construct the transfer matrix $A(\mathcal{S}, TC, m)$ for thin cylinders and $\vec{a}(\mathcal{S}, TC, m)$, $\vec{b}(\mathcal{S}, TC, m)$, such that

$$f(\mathcal{S}, TC, m; n) = \vec{a}(\mathcal{S}, TC, m)\, A^n(\mathcal{S}, TC, m)\, \vec{b^t}(\mathcal{S}, TC, m). \tag{2.4}$$

We conclude by showing how to construct new $\vec{a}(\mathcal{S}, T, m)$, for the torus such that

$$f(\mathcal{S}, T, m; n) = \vec{a}(\mathcal{S}, T, m)\, A^n(\mathcal{S}, TC, m)\, \vec{b^t}(\mathcal{S}, TC, m)$$
(2.5)

(so the transfer matrix and $\vec{b}$ for thin-cylinders and tori will be the same).

2.1 Specifics Fix m, the height of the grid. The technique starts by defining *legal objects* in $G(n, m)$ and letting $\mathcal{L}(n, m)$ be the set of legal objects in $G(n, m)$. The structures that we are counting must be legal objects but there may be many other legal objects as well.

For **ST** *(spanning trees), legal objects will be forests in (n, m) having the property that every connected component of the forest contains at least one vertex in $L(m) \cup R(n, m)$. See Figure 3.*

The next step is to define a set of *classifications* $\mathcal{P}$ of legal objects. The classifications will be expressed in terms of the $2m$ elements in $L(m) \cup R(n, m)$. Every $L \in \mathcal{L}(n, m)$ will have a unique classification $C(L) \in \mathcal{P}$. Set $\mathcal{L}_X(n, m) = \{L \in \mathcal{L}(n, m) : C(L) = X\}$ and $f_X(n) = |\mathcal{L}_X(n, m)|$. (Note that in $C(L)$, n should be considered a *label* and not a value.)

Classifications have the property that if L is a $\mathcal{S}$ structure with $C(L) = X$ then *every* legal object L' with $C(L') = X$ must also be a good $\mathcal{S}$ structure.

Order the elements of $\mathcal{P}$ arbitrarily as $X_1, X_2, \ldots, X_{|\mathcal{P}|}$. We then set $\vec{f}(n)$ to be the $|\mathcal{P}|$-tuple $\vec{f}(n) = (f_{X_1}(n), f_{X_2}(n), \ldots, f_{X_{|\mathcal{P}|}}(n))$.

See Figure 3. For **ST***, $\mathcal{P}$ will be the set of partitions of $2m$ elements in $L(m) \cup R(n, m)$. The classification of legal forest F in $G(n, m)$ will be the partition of $L(m) \cup R(n, m)$ such that $x, y \in L(m) \cup R(n, m)$ are in the same set in the partition $C(F)$ if and only if x, y are in the same connected component of F.*

Note that in a spanning tree every node is in the same connected component so every spanning tree has the same classification which is the partition containing the one set $X = \{L(m) \cup R(n, m)\}$. Furthermore every legal object with classification $X = \{L(m) \cup R(n, m)\}$ is a spanning tree.

The next step is to show that legal structures with the same classification behave the same when the same set of edges are added/subtracted from them. This is encapsulated in the following properties.

(P1) Let $E \subseteq \texttt{RtG}(n, m)$ and $X \in \mathcal{P}$. $\forall L \in \mathcal{L}_X(n, m)$ either all $L \cup E$ are legal objects in $\mathcal{L}(n+1, m)$ or no $L \cup E$ are legal objects in $\mathcal{L}(n+1, m)$. We denote these options by $X \cup E \neq \emptyset$ or $X \cup E = \emptyset$

(P2) Let $E \subseteq \texttt{RtG}(n, m)$ and $X \in \mathcal{P}$. If $X \cup E \neq \emptyset$ then there exists a unique $X' \in \mathcal{P}$ such that $\forall L \in \mathcal{L}_X(n, m)$ $C(L \cup E) = X'$. We denote this by $X \cup E = X'$.

(P3) If L is a legal object in $\mathcal{L}(n + 1, m)$, let $L - \texttt{RtG}(n, m)$ be the object in $G(n, m)$ that is created by starting with L and throwing away all of the edges in $\texttt{RtG}(n, m)$ and vertices in $R(n, m)$. Property (P3) is that $L - \texttt{RtG}(n, m)$ must be a legal object in $\mathcal{L}(n+1, m)$.

For spanning trees these properties are immediately obvious. (P1) and (P2) says that if two legal spanning forests L_1, L_2 in $\mathcal{L}(n, m)$ induce the same partition on the vertices in $L(m) \cup R(n, m)$ then adding the same set of edges $E \subseteq \texttt{RtG}(n, m)$ to L_1, L_2 either creates legal forests of both or it doesn't (e.g., it causes a cycle in both). (P3) states that if L is a legal forest in $\mathcal{L}(n+1, m)$ then throwing away the edges in the last column and the edges connecting the last column to the column preceding it, leaves a legal forest in the smaller grid graph.

For $X, Y \in \mathcal{P}$ define

(2.6) $$a_{Y,X} = |\{E \subseteq \texttt{RtG}(n, m) : X \cup E = Y\}$$

to be the number of subsets of the "new edges" $\texttt{RtG}(n, m)$ that, added to a legal structure in $\mathcal{L}_X(n, m)$, yield a legal structure in $\mathcal{L}_Y(n+1, m)$,. Properties (P1), (P2) tell us that this is *independent* of the actual structure and n and only dependent upon X and E. Since $\mathcal{P}$ and $\texttt{RtG}(n, m)$ are finite (size dependent only upon $\mathcal{S}$, and m) these values can be calculated. (P3) tells us that all legal structures in $\mathcal{L}_Y(n + 1, m)$ are built from legal structures in $\mathcal{L}(n, m)$. Combining yields

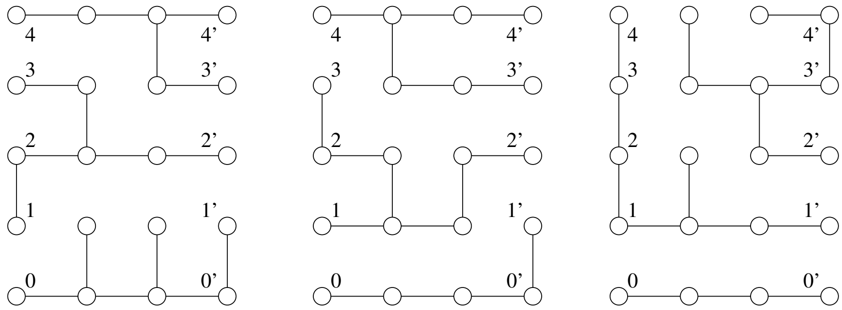
(2.7) $$\forall Y \in \mathcal{P},\ f_Y(n + 1) = \sum_{X \in \mathcal{P}} a_{Y,X} f_X(n)$$

Letting $A = \{a_{Y,X}\}_{X,Y \in \mathcal{P}}$ (where the ordering of the X, Y are the same as in $\vec{f}(n)$) this last equation can be rewritten as

(2.8) $$\vec{f}(n + 1) = A(\vec{f}(n))^t \text{ or } \vec{f}(n + 1) = A^n(\vec{f}(1))^t$$

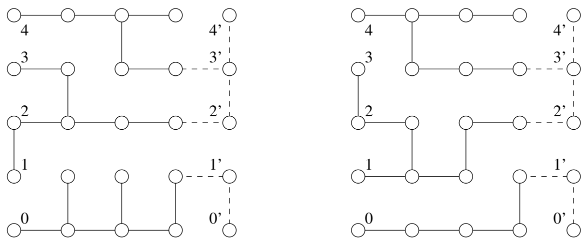
To finish, let $a_X = 1$ if legal objects of classification X are $\mathcal{S}$ structures and $a_X = 0$ otherwise. Set $\vec{a} = (a_{X_1}(n), a_{X_2}(n), \ldots, a_{X_{|\mathcal{P}|}}(n))$ and $\vec{b} = \vec{f}(1)$. Then

$$f(\mathcal{S}, G, m; n) = \sum_{X \in P} a_X f_X(n) = \vec{a}(\vec{f}(n))^t = \vec{a} A^n \vec{b^t}$$
(2.9)

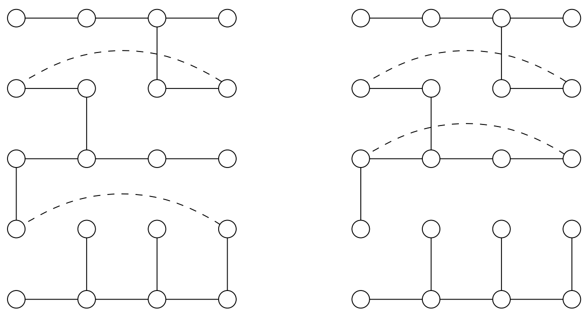


(a) A (b) B (c) C

Figure 3: Three legal forests in $G(4,5)$. $C(A) = C(B) = \{0,0',1'\}\{1,2,3,2'\}\{4,3',4'\}$ while $C(C) = \{0,0'\}\{1,2,3,4,1'\}\{3',4',5'\}$. For simplicity we are using i to denote $(0,i)$ and i' to denote $(n-1,i)$.



(a) $A \cup E$ (b) $B \cup E$



(c) $A \cup E'$ (d) $A \cup E''$

Figure 4: Since $C(A) = C(B)$, adding the same set of (dotted) edges from $\texttt{RtG}(n,m)$ to A and B creates a legal forest with the same classification, in this case $\{0,0',1'\}\{1,2,3,4,2',3',4'\}$. In (c) and (d) we have that $E', E'' \subset \texttt{Side}(4,5)$. We see that adding E' to A creates a spanning tree in $FC(4,5)$. Adding E'' to A does not create a spanning tree in $FC(4,5)$

so we have proven eq (2.2), i.e., we have derived a recurrence relation for spanning trees in a fixed-height grid-graph.

In order to derive eq (2.3), the recurrence relation for fixed-height fat-cylinders, we will need two more properties.

(P4) Let L be any $\mathcal{S}$-structure in fat-cylinder $FC(n,m)$. Then $L - \texttt{Side}(n,m)$ is a legal object in $G(n,m)$.

(P5) Let $E \subseteq \texttt{Side}(n,m)$ and $X \in \mathcal{P}$. $\forall L \in \mathcal{L}_X(n,m)$ either all $L \cup E$ are $\mathcal{S}$-structures in $FC(n,m)$, in which case we say that (X,E) is *good*, or *no* $L \cup E$ is a legal $\mathcal{S}$-structures in $FC(n,m)$, in which case (X,E) is *bad*.

For spanning trees of fat-cylinders property (P4) is immediately obvious. The endpoints of the $\texttt{Side}(n,m)$ *edges are all in* $L(m) \cup R(n,m)$ *so all of the connected components that arise after disposing of the* $\texttt{Side}(n,m)$ *edges must contain at least one vertex in* $L(m) \cup R(n,m)$. *Property (P5) follows from similar observations.*

From (P5) we can define

(2.10) $a_X = |\{E \subseteq \texttt{Side}(n,m) \,:\, (X,E) \text{ is good}\}|$

to be the number of subsets of $\texttt{Side}(n,m)$ that can make a X-legal structure in $G(n,m)$ into a spanning tree of $FC(n,m)$. From (P4)

$$f(\mathcal{S}, FC, m; n) = \sum_{X \in P} a_X f_X(n) = \vec{a}(\vec{f}(n))^t = \vec{a}A^n\vec{b}^t \tag{2.11}$$

and we have proven eq (2.3).

We have just seen how to count structures for grid graphs and fat-cylinders. This essentially follows from the facts that $E_G(n+1,m) = E_G(n,m) \cup \texttt{RtG}(n,m)$ and $E_{FC}(n,m) = E_G(n,m) \cup \texttt{Side}(n,m)$

The technique to count structures in thin-cylinders and tori is almost *exactly the same*. The only differences arise from the facts that

$$E_{TC}(n+1,m) \quad = \quad E_{TC}(n,m) \cup \texttt{RtTC}(n,m)$$

and

$$E_T(n,m) \quad = \quad E_{TC}(n,m) \cup \texttt{Side}(n,m),$$

so we must replace $\texttt{RtG}(n,m)$ in our properties by

$$\texttt{RtTC}(n,m) = \texttt{RtG}(n,m) \cup \{((m-1,n),(0,n))\},$$

e.g., in (P1), (P2), (P3), and replace $FC(n,m)$ by $T(n,m)$ in P(4) and (P5). We then need to check that all properties still hold, which they do in all of our cases. Note that we must also change $\texttt{RtG}(n,m)$ to $\texttt{RtTC}(n,m)$ in eq (2.6). This changes the values of the $a_{X,Y}$ which, in turn, changes the transfer matrix $A = \{a_{X,Y}\}$. After making these changes we then derive eq (2.4) and (2.5).

For spanning trees it is easy to see that, using the same definitions of legal objects and classifications, properties (P1)-(P5) still hold for thin-cylinders and tori so the derivations of the number of spanning trees in fixed-height thin-cylinders and tori remain correct.

3 Comments and Extensions

In this note we sketched the technique for using transfer matrices to count various types of structures in grid graphs, cylinders, and tori. Although this technique has been widely used in various forms for grid-graphs it doesn't seem to have been previously explicitly described for the other types of graphs. One possible reason for this lack, is that while most of the papers referenced in table 1 did use transfer matrices, because they were only interested in the grid graph they only indexed their classifications using the nodes $R(n,m)$ (from the right-hand side) and *did not* use the nodes $L(m)$ (from the left hand side). This suffices for the grid case and actually leads to a smaller transfer matrix. But, to count structures in fat-cylinders or tori, it is necessary to understand how $\texttt{Side}(m.n)$ contributes and, to do that, it is necessary to index the states using $L(m) \cup R(n,m)$.

We should point out that the technique in this paper, by its very generality, is by necessity not particularly efficient. For *specific* structures it is usually possible to improve the calculations. As an example, in the case of Hamiltonian cycles on grid graphs Stoyan and Strehl [17] showed that the fact that the Hamiltonian cycle can not cross itself tremendously reduces the size of what we call $\mathcal{P}$ (by showing a correspondence between *achievable* classifications and Motzkin words). Another way of reducing complexity is by showing that the associated transfer matrix has a specific structure [19], e.g., block diagonalizable with very special blocks, that reduces the size of its characteristic polynomial.

Finally, we point out that the technique described here would also work to count structures in "Mobius Cylinders". These are the $G(n,m)$ grid graph where the ends are connected together with a twist, i.e, instead of adding $\texttt{Side}(n,m) = \{((n,i),\,(0,i)) \,:\, 0 \le i < m\}$ to $G(n,m)$ we add $\texttt{Mobius}(n,m) = \{((n,i),\,(0,m-1-i)) \,:\, 0 \le i < m\}$. The only change needed in the analysis is to replace eq (2.10) by $a_X = |\{E \subseteq \texttt{Mobius}(n,m) \,:\, (X,E) \text{ is good}\}|$. The canonical example of this type of graph is the *Mobius ladder* which has $m = 2$. Thus, the techniques in this paper easily permit counting all types of structures on the Mobius ladder (which was recently done in a different way by [11]).

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A Appendix

As an illustration of our technique we have derived the recurrence relations for the number of spanning trees in the height-3 grids, fat cylinders, thin-cylinders and tori and appended them below. For technical reasons (simplifying calculations) we start our recurrence relations with initial values at $n = 3$.

1) $f(n) = \mathbf{ST}_G(n, 3)$, the number of spanning trees in the height-3 grid, satisfies,

$$\begin{aligned} f(n) &= 15f(n-1) - 32f(n-2) \\ &\quad +15f(n-3) - f(n-4) \end{aligned}$$

with initial values 192, 2415, 30305, 380160 for $n = 3, 4, 5, 6$

Calculation shows that $f(n) \sim 0.0975\ldots \times \phi_1^n$ where $\phi_1 = 12.543\ldots$.

2) $f(n) = \mathbf{ST}_{FC}(n, 3)$, the number of spanning trees in the height-3 fat-cylinder, satisfies,

$$\begin{aligned} T(n) &= 48T(n-1) - 960T(n-2) + 10622T(n-3) \\ &\quad -73248T(n-4) + 335952T(n-5) \\ &\quad -1065855T(n-6) + 2396928T(n-7) \\ &\quad -3877536T(n-8) + 4548100T(n-9) \\ &\quad -3877536T(n-10) + 2396928T(n-11) \\ &\quad -1065855T(n-12) + 335952T(n-13) \\ &\quad -73248T(n-14) + 10622T(n-15) \\ &\quad -960T(n-16) + 48T(n-17) - T(n-18) \end{aligned}$$

with initial values 1728, 31500, 508805, 7741440, 113742727, 1633023000, 23057815104, 321437558750, 4435600730891, 60699082752000, 824853763418893, 11142718668655210, 149755467741359040, 2003730198180606000, 26705200059067689617, 354688416147207905280, 4696298144208387062419, 62009696321724473437500 for $n = 3, 4, ..., 20$

Calculation shows that $f(n) \sim \frac{1}{3}n \times \phi_1^n$ where ϕ_1 is as above.

3) $f(n) = \mathbf{ST}_{TC}(n, 3)$, the number of spanning trees in the height-3 thin-cylinder, satisfies,

$$T(n) = 24T(n-1) - 24T(n-2) + T(n-3)$$

with initial values 1728, 39675, 910803 for $n = 3, 4, 5$

Calculation shows that $f(n) \sim 0.142\ldots \times \phi_2^n$ where $\phi_2 = 22.956\ldots$.

4) $f(n) = \mathbf{ST}_T(n, 3)$, the number of spanning trees in the height-3 torus, satisfies,

$$\begin{aligned} T(n) &= 58T(n-1) - 1131T(n-2) + 8700T(n-3) \\ &\quad -29493T(n-4) + 43734T(n-5) \\ &\quad -29493T(n-6) + 8700T(n-7) \\ &\quad -1131T(n-8) + 58T(n-9) - T(n-10) \end{aligned}$$

with initial values 11664, 367500, 10609215, 292626432, 7839321861, 205683135000, 5312031978672, 135495143785470, 3421536337406913, 85686871818240000 for $n = 3, 4, ..., 12$

Calculation shows that $f(n) \sim \frac{1}{3}n \times \phi_2^n$ where ϕ_2 is as above.

Note: We can rewrite the above as

$$\begin{aligned} \mathbf{ST}_G(n,3) &\sim c_1\phi_1^n, & \mathbf{ST}_{FC}(n,3) &\sim c_1' n\phi_1^n, \\ \mathbf{ST}_{TC}(n,3) &\sim c_2\phi_2^n, & \mathbf{ST}_T(n,3) &\sim c_2' n\phi_2^n \end{aligned}$$

for appropriate constants $c_1, c_1'.c_2, c_2', \phi_1, \phi_2$. It is not surprising that $\mathbf{ST}_G(n,3)$ and $\mathbf{ST}_{FC}(n,3)$ share the same first-order growth rate ϕ_1^n; ϕ_1 is just the largest eigenvalue of the matrix $A(\mathbf{ST}, FC, m) = A(\mathbf{ST}, G, m)$ which defines both of them. Similarly, ϕ_2 is the largest eigenvalue of the matrix $A(\mathbf{ST}, TC, m) = A(\mathbf{ST}, T, m)$. What is a-priori unexpected is the fact that $\mathbf{ST}_G(n,3)$ and $\mathbf{ST}_{TC}(n,3)$ both grow as ϕ_i^n but $\mathbf{ST}_{FC}(n,3)$ and $\mathbf{ST}_T(n,3)$ both grow as $n\phi_i^n$ (for the associated i). It would be interesting to study whether this is just a coincidence or a reflection of some more general phenomenon.