Abstract

We investigate the impact the negative curvature has on the traffic congestion in large-scale networks. We prove that every Gromov hyperbolic network $G$ admits a core, thus answering in the positive a conjecture by Jonckheere, Lou, Bonahon, and Baryshnikov, Internet Mathematics, 7 (2011) which is based on the experimental observation by Narayan and Saniee, Physical Review E, 84 (2011) that real-world networks with small hyperbolicity have a core congestion. Namely, we prove that for every subset $X$ of $n$ vertices of a graph with $\delta$-thin geodesic triangles (in particular, of a $\delta$-hyperbolic graph) $G$ there exists a vertex $m$ of $G$ such that the ball $B(m, 4\delta)$ of radius $4\delta$ centered at $m$ intercepts at least one half of the total flow between all pairs of vertices of $X$, where the flow between two vertices $x, y \in X$ is carried by geodesic (or quasi-geodesic) $(x, y)$-paths. Moreover, we prove a primal-dual result showing that, for any commodity graph $R$ on $X$ and any $r \geq 8\delta$, the size $\sigma_r(R)$ of the least $r$-multi-core (i.e., the number of balls of radius $r$) intercepting all pairs of $R$ is upper bounded by the maximum number of pairwise $(2r - 5\delta)$-apart pairs of $R$ and that an $r$-multi-core of size $\sigma_{r-5\delta}(R)$ can be computed in polynomial time for every finite set $X$. Our result about total $r$-multi-cores is based on a Helly-type theorem for quasiconvex sets in $\delta$-hyperbolic graphs (this is our second main result). Namely, we show that for any finite collection $Q$ of pairwise intersecting $\epsilon$-quasiconvex sets of a $\delta$-hyperbolic graph $G$ there exists a single ball $B(c, 2\epsilon + 5\delta)$ intersecting all sets of $Q$. More generally, we prove that if $Q$ is a collection of $2r$-close (i.e., any two sets of $Q$ are at distance $\leq 2r$) $\epsilon$-quasiconvex sets of a $\delta$-hyperbolic graph $G$, then there exists a ball $B(c, r^*)$ of radius $r^* := \max\{2\epsilon + 5\delta, r + \epsilon + 3\delta\}$ intersecting all sets of $Q$. These kind of Helly-type results are also useful in geometric group theory. Using the Helly theorem for quasiconvex sets and a primal-dual approach, we show algorithmically that the minimum number of balls of radius $2\epsilon + 5\delta$ intersecting all sets of a family $Q$ of $\epsilon$-quasiconvex sets does not exceed the packing number of $Q$ (maximum number of pairwise disjoint sets of $Q$). We extend the covering and packing result to set-families $^kQ$ in which each set is a union of at most $\kappa$ $\epsilon$-quasiconvex sets of a $\delta$-hyperbolic graph $G$. Namely, we show that if $r \geq \epsilon + 2\delta$ and $\pi_r(^kQ)$ is the maximum number of mutually $2r$-apart members of $^kQ$, then the minimum number of balls of radius $r + 2\epsilon + 6\delta$ intersecting all members of $^kQ$ is at most $2\kappa^2\pi_r(^kQ)$ and such a hitting set and a packing can be constructed in polynomial time for every finite $^kQ$ (this is our third main result). For set-families consisting of unions of $\kappa$ balls in $\delta$-hyperbolic graphs a similar result was obtained by Chepoi and Estellon (2007). In case of $\delta = 0$ (trees) and $\epsilon = r = 0$, (subtrees of a tree) we recover the result of Alon (2002) about the transversal and packing numbers of a set-family in which each set is a union of at most $\kappa$ subtrees of a tree.