Abstract

Separating decompositions of metric spaces are an important randomized clustering paradigm that was formulated by Bartal in (*Probabilistic approximation of metric spaces and its algorithmic applications*, FOCS '96 184–193, 1996) and is defined as follows. Given a metric space (X, d_X), its *modulus of separated decomposability*, denoted $\mathsf{SEP}(X, d_X)$, is the infimum over those $\sigma \in (0, \infty]$ such that for every finite subset $S \subseteq X$ and every $\Delta > 0$ there exists a distribution over random partitions $\mathcal{P}$ of S into sets of diameter at most Δ such that for every $x, y \in S$ the probability that both x and y do not fall into the same cluster of the random partition $\mathcal{P}$ is at most $\sigma d_X(x, y)/\Delta$. Here we obtain new bounds on $\mathsf{SEP}(X, \|\cdot\|_X)$ when $(X, \|\cdot\|_X)$ is a finite dimensional normed space, yielding, as a special case, that $\sqrt{n} \lesssim \mathsf{SEP}(\ell_\infty^n) \lesssim \sqrt{n \log n}$ for every $n \in \mathbb{N}$. More generally, $\sqrt{n} \lesssim \mathsf{SEP}(\ell_p^n) \lesssim \sqrt{n \min\{p, \log n\}}$ for every $p \in [2, \infty]$. This improves over the work of Charikar, Chekuri, Goel, Guha, and Plotkin (*Approximating a finite metric by a small number of tree metrics*, FOCS '98, 379–388, 1998), who obtained this bound when $p = 2$, yet for $p \in (2, \infty]$ they obtained the asymptotically weaker estimate $\mathsf{SEP}(\ell_p^n) \lesssim n^{1-1/p}$. One should note that it was claimed in (M. Charikar *et al.*, FOCS '98) that the bound $\mathsf{SEP}(\ell_p^n) \lesssim n^{1-1/p}$ is sharp for every $p \in [2, \infty]$, and in particular it was claimed in (M. Charikar *et al.*, FOCS '98) that $\mathsf{SEP}(\ell_\infty^n) \asymp n$. However, the above results show that this claim of (M. Charikar *et al.*, FOCS '98) is incorrect for every $p \in (2, \infty]$. Our new bounds on the modulus of separated decomposability rely on extremal results for orthogonal hyperplane projections of convex bodies, specifically using the work of Barthe and the author in (F. Barthe and A. Naor, *Hyperplane projections of the unit ball of ℓ_p^n*, Discrete Comput. Geom., 27(2):215–226, 2002). This yields additional refined estimates, an example of which is that for every $n \in \mathbb{N}$ and $k \in \{1, \ldots, n\}$ we have $\mathsf{SEP}\big((\ell_2^n)_{\leqslant k}\big) \lesssim \sqrt{k \log(en/k)}$, where $(\ell_2^n)_{\leqslant k}$ denotes the subset of $\mathbb{R}^n$ consisting of all those vectors that have at most k nonzero entries, equipped with the Euclidean metric. The above statements have implications to the Lipschitz extension problem through its connection to random partitions that was developed by Lee and the author in (J. R. Lee and A. Naor, *Extending Lipschitz functions via random metric partitions*, Invent. Math., 160(1):59–95, 2005). Given a metric space (X, d_X), let $\mathsf{e}(X)$ denote the infimum over those $K \in (0, \infty]$ such that for every Banach space Y and every subset $S \subset X$, every 1-Lipschitz function $f : S \to Y$ has a K-Lipschitz extension to all of X. Johnson, Lindenstrauss and Schechtman proved in (*Extensions of Lipschitz maps into Banach spaces*, Israel J. Math., 54(2):129–138, 1986) that $\mathsf{e}(X) \lesssim \dim(X)$ for every finite dimensional normed space $(X, \|\cdot\|_X)$. It is a longstanding open problem to determine the correct asymptotic dependence on $\dim(X)$ in this context, with the best known lower bound, due to Johnson and Lindenstrauss (*Extensions of Lipschitz mappings into a Hilbert space*, in Conference in Modern Analysis and Probability (New Haven, Conn., 1982), volume 26 of Contemp. Math., pages 189–206. Amer. Math. Soc., 1984), being that the quantity $\mathsf{e}(X)$ must sometimes be at least a constant multiple of $\sqrt{\dim(X)}$. In particular, the previously best known upper bound on $\mathsf{e}(\ell_\infty^n)$ was the $O(n)$ estimate of (W. B. Johnson *et al.*, 1986). It is shown here that for every $n \in \mathbb{N}$ we have $\sqrt{n} \lesssim \mathsf{e}(\ell_\infty^n) \lesssim \sqrt{n \log n}$, thus answering (up to logarithmic factors) a question that was posed by Brudnyi and Brudnyi as Problem 2 in (A. Brudnyĭ and Y. Brudnyĭ, *Simultaneous extensions of Lipschitz functions*, Uspekhi Mat. Nauk, 60(6(366)):53–72, 2005). More generally, $\mathsf{e}(\ell_p^n) \lesssim \sqrt{n \min\{p, \log n\}}$ for every $p \in [2, \infty]$, thus resolving (negatively) a conjecture that Brudnyi and Brudnyi posed as Conjecture 5 in (A. Brudnyĭ and Y. Brudnyĭ, 2005).