

## Abstract

We consider the problem of solving *systems of multivariate polynomial equations of degree  $k$*  over a finite field. For every integer  $k \geq 2$  and finite field  $\mathbb{F}_q$  where  $q = p^d$  for a prime  $p$ , we give, to the best of our knowledge, the first algorithms that achieve an exponential speedup over the brute force  $O(q^n)$  time algorithm in the worst case. We present two algorithms, a randomized algorithm with running time  $q^{n+o(n)} \cdot q^{-n/O(k)}$  time if  $q \leq 2^{4ekd}$ , and  $q^{n+o(n)} \cdot (\frac{\log q}{dek})^{-dn}$  otherwise, where  $e = 2.718\dots$  is Napier's constant, and a deterministic algorithm for *counting* solutions with running time  $q^{n+o(n)} \cdot q^{-n/O(kq^{6/7d})}$ . For the important special case of quadratic equations in  $\mathbb{F}_2$ , our randomized algorithm has running time  $O(2^{0.8765n})$ . For systems over  $GF(2)$  we also consider the case where the input polynomials do not have bounded degree, but instead can be efficiently represented as a  $\Sigma\Pi\Sigma$  circuit, i.e., a sum of products of sums of variables. For this case we present a deterministic algorithm running in time  $2^{n-\delta n}$  for  $\delta = 1/O(\log(s/n))$  for instances with  $s$  product gates in total and  $n$  variables. Our algorithms adapt several techniques recently developed via the polynomial method from circuit complexity. The algorithm for systems of  $\Sigma\Pi\Sigma$  polynomials also introduces a new *degree reduction* method that takes an instance of the problem and outputs a subexponential-sized set of instances, in such a way that feasibility is preserved and every polynomial among the output instances has degree  $O(\log(s/n))$ .