Abstract

Permutation Pattern Matching (or PPM) is a decision problem whose input is a pair of permutations $\pi$ and $\tau$, represented as sequences of integers, and the task is to determine whether $\tau$ contains a subsequence order-isomorphic to $\pi$. Bose, Buss and Lubiw proved that PPM is NP-complete on general inputs. We show that PPM is NP-complete even when $\pi$ has no decreasing subsequence of length 3 and $\tau$ has no decreasing subsequence of length 4. This provides the first known example of PPM being hard when one or both of $\pi$ and $\tau$ are restricted to a proper hereditary class of permutations. This hardness result is tight in the sense that PPM is known to be polynomial when both $\pi$ and $\tau$ avoid a decreasing subsequence of length 3, as well as when $\pi$ avoids a decreasing subsequence of length 2. The result is also tight in another sense: we will show that for any hereditary proper subclass $C$ of the class of permutations avoiding a decreasing sequence of length 3, there is a polynomial algorithm solving PPM instances where $\pi$ is from $C$ and $\tau$ is arbitrary. We also obtain analogous hardness and tractability results for the class of so-called skew-merged patterns. From these results, we deduce a complexity dichotomy for the PPM problem restricted to $\pi$ belonging to $\text{Av}(\alpha)$, where $\text{Av}(\alpha)$ denotes the class of permutations avoiding a permutation $\alpha$. Specifically, we show that the problem is polynomial when $\alpha$ is in the set $\{1, 12, 21, 132, 213, 231, 312\}$, and it is NP-complete for any other $\alpha$. 