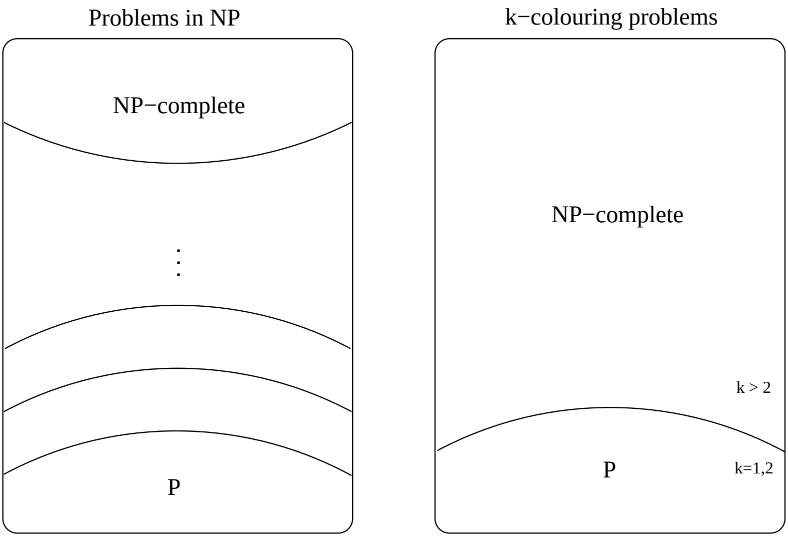
Constraint Satisfaction and Graph Theory

Pavol Hell

SIAM DM, June 2008

NP versus Colouring



Constraint Satisfaction Problems

CSP

Assign values to variables so that all constraints are satisfied

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Examples

- SAT

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- SAT
- 3-COL

Constraint Satisfaction Problems

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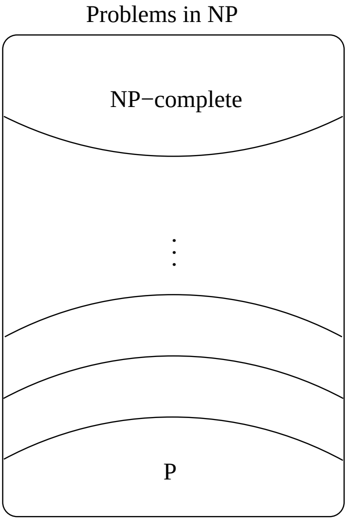
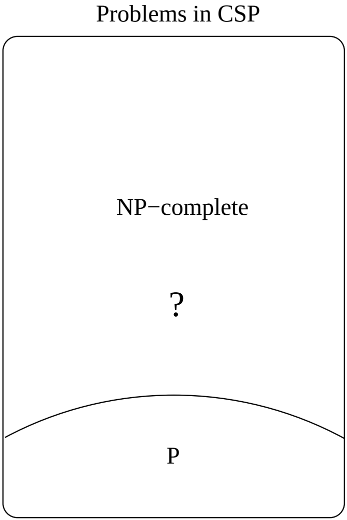
Assign values to variables so that all constraints are satisfied

Examples

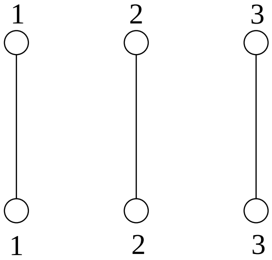
- SAT
- 3-COL
- $(x, y) \in \{(1, 1), (2, 3)\}$ and $(x, z, w) \in \{(2, 2, 1), (1, 3, 2), (2, 2, 2)\}$

NP versus CSP





Generalizing Colouring



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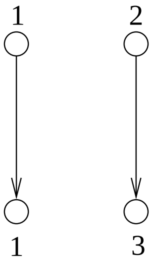

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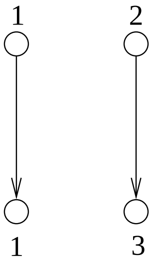

Which Problems / Complexities

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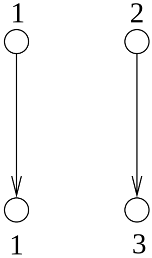
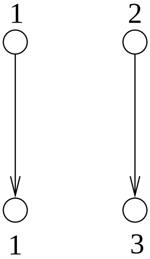


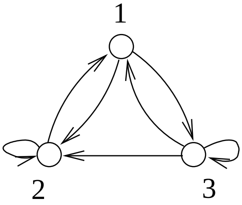
Which Problems / Complexities

What is this problem?

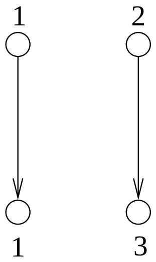
Generalizing Colouring



= HOM(H)



H



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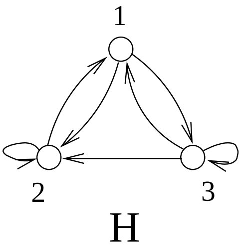


Homomorphism problem HOM(H)

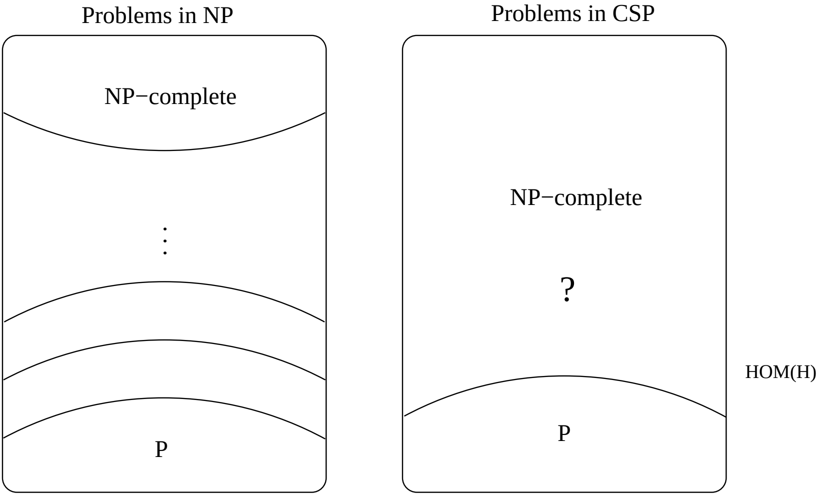
A colouring of a graph G without the above pattern is exactly a homomorphism to H

Generalizing Colouring

Feder-Vardi

Each constraint satisfaction problem is polynomially equivalent to HOM(H) for some digraph H

NP versus CSP



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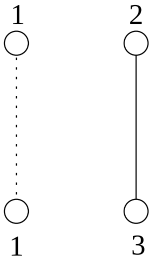

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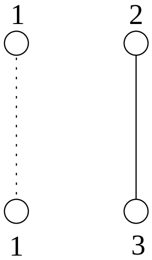
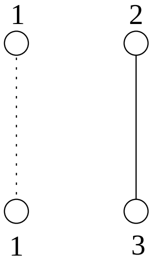


Which Problem

Clique Cutset Problem

Generalizing Colouring

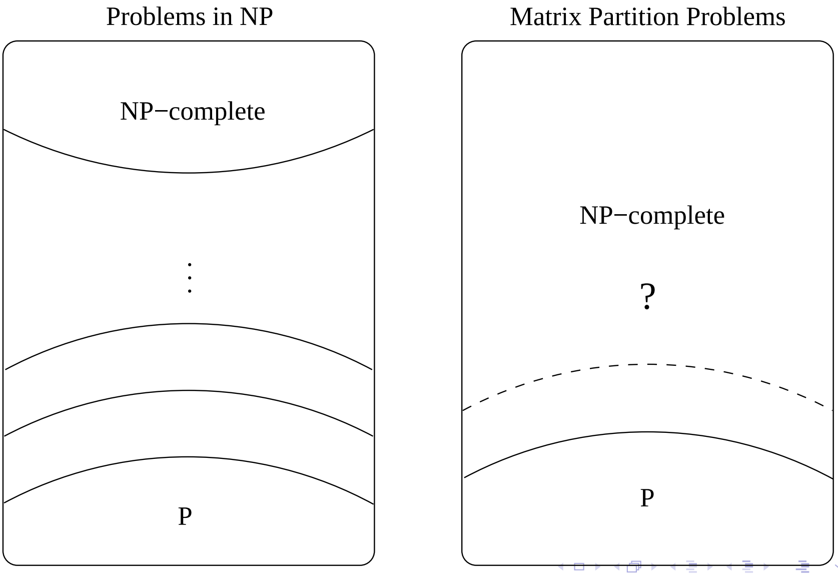


Which Problem

Clique Cutset Problem

Matrix partition problems (Feder-Hell-Motwani-Klein)

NP versus MPP



Suspicious Problems

Cameron-Eschen-Hoang-Sritharan

Stubborn problem

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Given a 3-edge-coloured K_n, colour the vertices without a monochromatic edge

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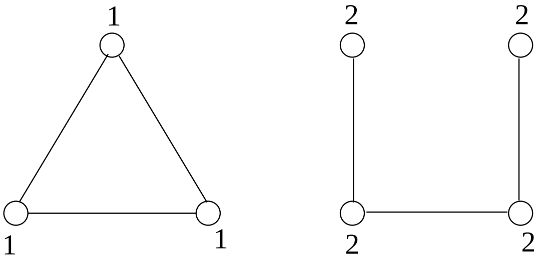
Complexity ? (Feder-Hell-Kral-Sgall)

Generalizing Colouring



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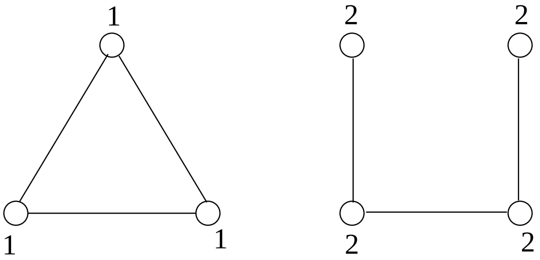

Which Problem

Is H partitionable into a triangle-free graph and a cograph?

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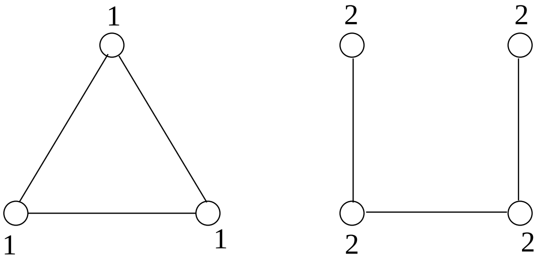
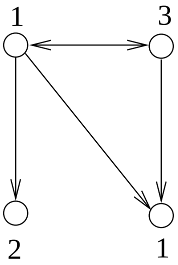


Which Problem

Is H partitionable into a triangle-free graph and a cograph?

Partition problems, generalized colouring problems, subcolouring problems, etc., Alekseev, Farrugia, Lozin, Broersma, Fomin, Nesetril, Woeginger, Ekim, de Werra, Stacho, MacGillivray, Yu, Hoang, Le, etc.

Generalizing Colouring



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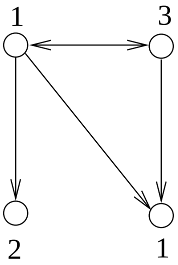

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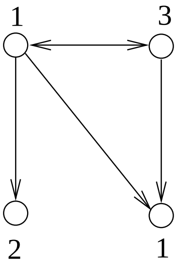

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Fagin + Feder-Vardi + Kun-Nesetril + Nesetril-Tardif

NP versus CSP

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Fagin + Feder-Vardi + Kun-Nesetril + Nesetril-Tardif

- Every problem in NP is polynomially equivalent to a pattern-forbidding colouring problem

NP versus CSP

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- Every problem in CSP is polynomially equivalent to a pattern-forbidding colouring problem with patterns on single edges

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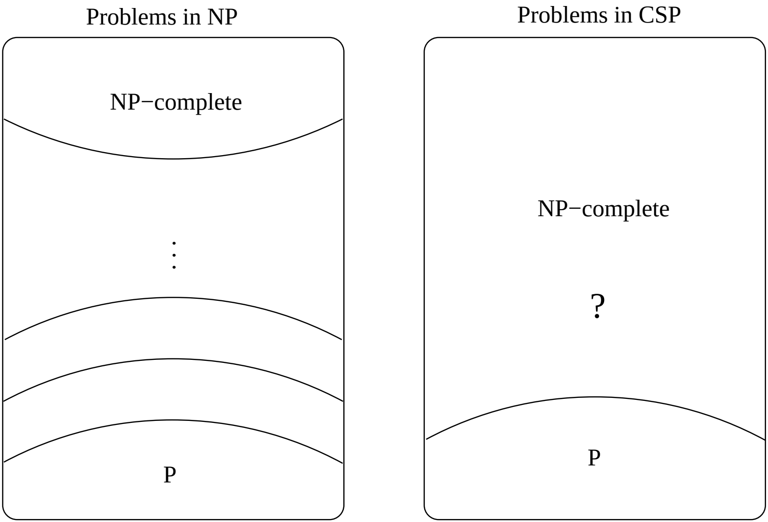
Some finite set of patterns corresponds to isomorphism complete problems. What does it look like?

NP versus CSP



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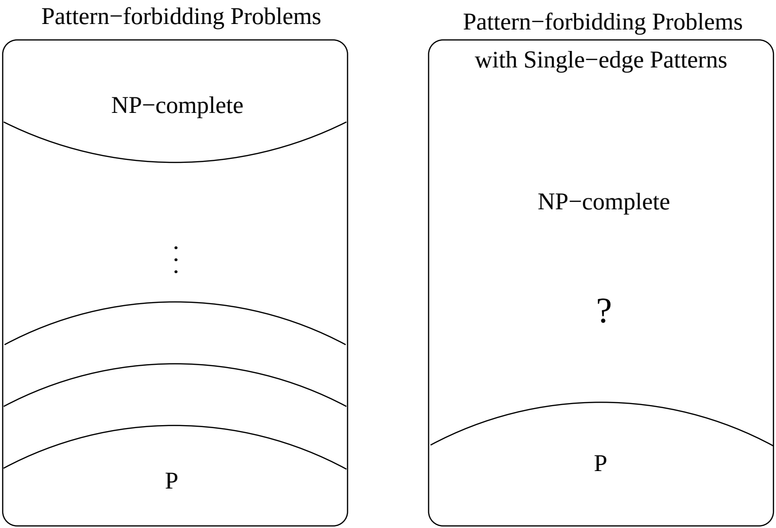
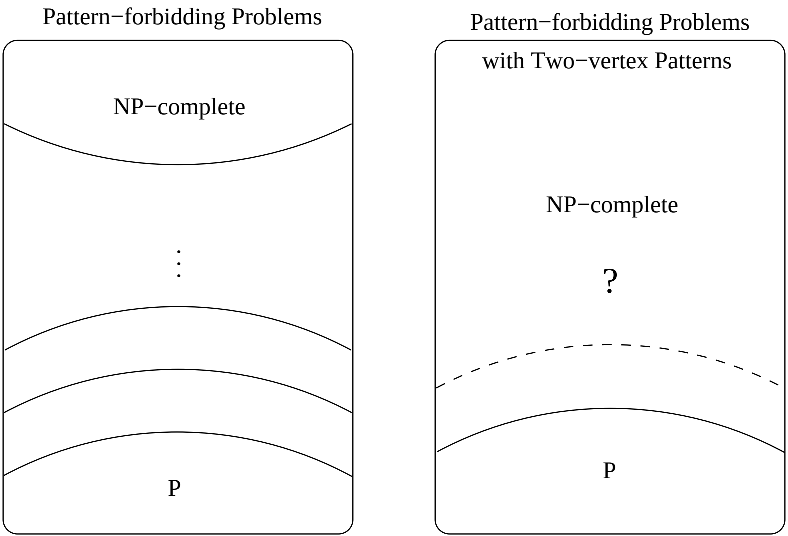


NP versus MPP



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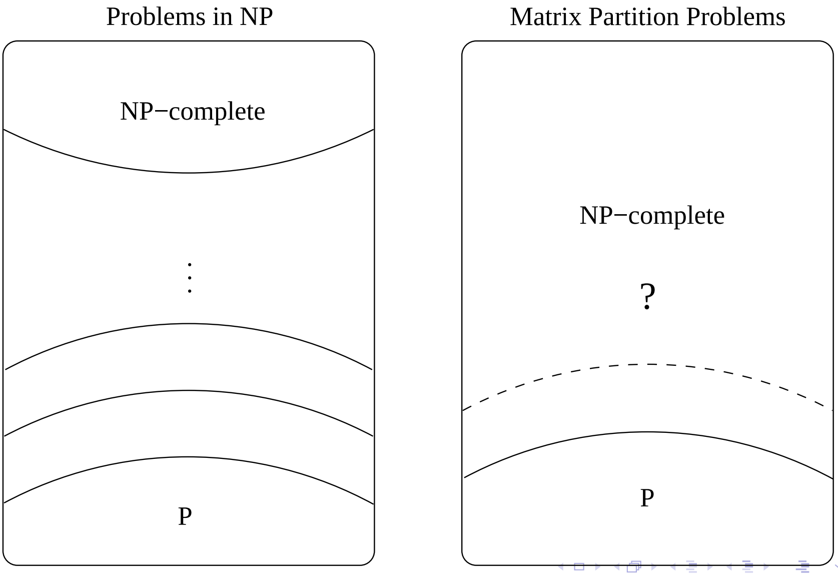

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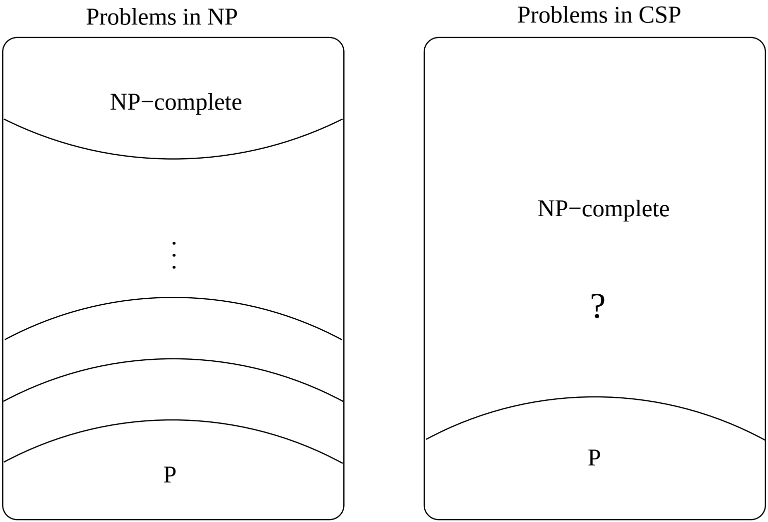

Polymorphisms

POL(H) for a digraph H

$f : V(H)^k \to V(H)$ such that
$a_i b_i \in E(H) \ \forall i \implies f(a_1, a_2, \ldots, a_k) f(b_1, b_2, \ldots, b_k) \in E(H)$.

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Jeavons

If POL(H) $\subseteq$ POL(H'), then HOM(H') reduces to HOM(H)

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Jeavons

If POL(H) $\subseteq$ POL(H'), then HOM(H') reduces to HOM(H)

The more polymorphisms H has, the more likely is HOM(H) to be polynomial

How small can POL(H) be?

Projective H

POL(H) consists only of all projections, composed with automorphisms of H

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K_n

The graph K_n, with $n \geq 3$, is projective

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Projective H

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K_n

The graph K_n, with $n \geq 3$, is projective

Therefore

If H is projective, then HOM(H) is NP-complete

Example Polymorphisms

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- Majority polymorphism:
  $f(u, u, v) = f(u, v, u) = f(v, u, u) = u$

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  $f(u, u, v) = f(u, v, u) = f(v, u, u) = u$
- Near unanimity polymorphism:
  $f(v, u, \dots, u) = \cdots = f(u, u, \dots, v) = u$

Examples

Example Polymorphisms

- Majority polymorphism:
  $f(u,u,v) = f(u,v,u) = f(v,u,u) = u$
- Near unanimity polymorphism:
  $f(v,u,\dots,u) = \cdots = f(u,u,\dots,v) = u$
- Weak unanimity polymorphism:
  $f(u,u,\dots,u) = u, \quad f(v,u,\dots,u) = \cdots = f(u,u,\dots,v)$

Examples

Example Polymorphisms

- Majority polymorphism:
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- Near unanimity polymorphism:
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- Weak unanimity polymorphism:
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- Maltsev polymorphism: $f(u,u,v) = f(v,u,u) = v$

Polynomial Algorithms

Feder-Vardi, Jeavons, Bulatov, Bulatov-Dalmau

If H has a near unanimity polymorphism, or a Maltsev polymorphism, then the problem HOM(H) is in P

Polynomial Algorithms

Universal Tool

All known polynomial cases are attributable to some nice polymorphism.

Feder-Vardi, Bulatov-Jeavons-Krokhin, Maroti-McKenzie, Nesetril-Siggers

HOM(H) is NP-complete if

Feder-Vardi, Bulatov-Jeavons-Krokhin, Maroti-McKenzie, Nesetril-Siggers

HOM(H) is NP-complete if

- H has no weak near unanimity polymorphism

Feder-Vardi, Bulatov-Jeavons-Krokhin, Maroti-McKenzie, Nesetril-Siggers

HOM(H) is NP-complete if

- H has no weak near unanimity polymorphism
- H is K_3-partitionable

Feder-Vardi, Bulatov-Jeavons-Krokhin, Maroti-McKenzie, Nesetril-Siggers

HOM(H) is NP-complete if

- H has no weak near unanimity polymorphism
- H is K_3-partitionable
- H is block-projective

Feder-Vardi, Bulatov-Jeavons-Krokhin, Maroti-McKenzie, Nesetril-Siggers

HOM(H) is NP-complete if

- H has no weak near unanimity polymorphism
- H is K_3-partitionable
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- some algebra in $VAR((V(H),\text{POL}(H))$ is projective,

Feder-Vardi, Bulatov-Jeavons-Krokhin, Maroti-McKenzie, Nesetril-Siggers

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- **and all these conditions are equivalent**

Feder-Vardi, Bulatov-Jeavons-Krokhin, Maroti-McKenzie, Nesetril-Siggers

HOM(H) is NP-complete if

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Conjecture

In all other cases HOM(H) can be solved in polynomial time

Feder-Vardi + Bulatov-Jeavons-Krokhin + Larose-Zadori

The Dichotomy Conjecture

The problem HOM(H) is

Feder-Vardi + Bulatov-Jeavons-Krokhin + Larose-Zadori

The Dichotomy Conjecture

The problem HOM(H) is

- In P is H admits a weak near unanimity polymorphism

Feder-Vardi + Bulatov-Jeavons-Krokhin + Larose-Zadori

The Dichotomy Conjecture

The problem HOM(H) is

- In P is H admits a weak near unanimity polymorphism
- NP-complete if H does not admit a weak near unanimity polymorphism

Example application:

Bang-Jensen - Hell Conjecture (1990)

Example application:

Barto-Kozik-Niven 2008

If H has neither sources nor sinks, then

- HOM(H) is in P if H retracts to a cycle
- HOM(H) is NP-complete otherwise

Bang-Jensen - Hell Conjecture (1990)

Example application:

Barto-Kozik-Niven 2008

If H has neither sources nor sinks, then

- HOM(H) is in P if H retracts to a cycle
- HOM(H) is NP-complete otherwise
  (there is no weak near unanimity polymorphism)

Reflexive Graphs

The following are equivalent

- G has a majority polymorphism

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- G is a retract of a product of paths

Graphs with Polymorphisms

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Hell-Rival, Nowakowski-Rival, Pesch-Poguntke

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Hell-Rival, Nowakowski-Rival, Pesch-Poguntke
Feder-Vardi - decidable in polynomial time

Brewster-Feder-Hell-Huang-MacGillivray 2008

Reflexive Graphs

- Any chordal graph G admits a near-unanimity polymorphism

Brewster-Feder-Hell-Huang-MacGillivray 2008

Reflexive Graphs

- Any chordal graph G admits a near-unanimity polymorphism
- A chordless cycle of length >3 does not admit a near-unanimity polymorphism

Larose-Loten-Zadori 2005

Reflexive Graphs

The following statements are equivalent

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Larose-Loten-Zadori 2005

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The following statements are equivalent

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- G^2 dismantles to the diagonal

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Brewster-Feder-Hell-Huang-MacGillivray 2008

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Brewster-Feder-Hell-Huang-MacGillivray 2008

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Brewster-Feder-Hell-Huang-MacGillivray 2008

Reflexive Graphs

The following statements are equivalent

- G has a conservative near-unanimity polymorphism $(f(a, b, \dots) \in \{a, b, \dots\})$
- G is an interval graph

Hell-Nesetril-Zhu, Feder-Vardi, Dalmau-Krokhin-Larose

For a connected H the following are equivalent

- H admits a near-unanimity operation

General Digraphs

Hell-Nesetril-Zhu, Feder-Vardi, Dalmau-Krokhin-Larose

For a connected H the following are equivalent

- H admits a near-unanimity operation
- H has bounded treewidth duality

General Digraphs

Hell-Nesetril-Zhu, Feder-Vardi, Dalmau-Krokhin-Larose

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- HOM(H) can be expressed in Datalog

General Digraphs

Hell-Nesetril-Zhu, Feder-Vardi, Dalmau-Krokhin-Larose

For a connected H the following are equivalent

- H admits a near-unanimity operation
- H has bounded treewidth duality
- HOM(H) can be expressed in Datalog
- RET(H) can be expressed by a first order formula

Finite duality

Rossman, Atserias, Larose-Loten-Zadori

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Nesetril-Tardif

If H has finite duality then H has tree duality