Irreducible Powerful Ray Patterns*

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1 Notations and Preliminaries

To explore the qualitative or combinatorial properties of nonnegative matries, many authors made use of Boolean matrix. And as a natural generalization of Boolean matrix, many authors considered sign pattern. *Sign pattern* is a matrix each of whose entries is $0, -1$ or 1 with its own algebra(See [2]). Sign pattern can be considered as abstraction of real matrix. So it is natural to consider abstraction of complex matrix. The authors of the recent paper(See [3]) studied this topic. *Ray pattern* is a matrix each of whose entries is either 0 or a ray $e^{i\theta}$ where θ is real number. Table 1 shows the addition and the multiplication of 0 and rays.

Table 1. *Addition and multiplication of 0 and rays*

$+$	$e^{i\theta_1}$	0	#
$e^{i\theta_2}$	$e^{i\theta_1}$ if $e^{i\theta_1} = e^{i\theta_2}$ # if $e^{i\theta_1} \neq e^{i\theta_2}$	$e^{i\theta_2}$	#
0	$e^{i\theta_1}$	0	#
#	#	#	#

$\cdot$	$e^{i\theta_1}$	0	#
$e^{i\theta_2}$	$e^{i(\theta_1+\theta_2)}$	0	#
0	0	0	0
#	#	0	#

We denote by # any sum of rays where at least two of the rays are distinct. And we call # ambiguous entry. The product of $m \times p$ ray pattern $A = [a_{ij}]$ and $p \times n$ ray pattern $B = [b_{ij}]$ is defined as usual; the (s,t) entry of AB is $\sum_{k=1}^{p} a_{sk} b_{kt}$. Note that the product of two ray patterns does not always yield ray pattern, since some entries of the product can be #.

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Let $A=(a_{rs})$ be an $n\times n$ ray pattern. The *digraph of* A , denoted $D(A)$, is the digraph with vertex set $\{1,2,\cdots,n\}$ such that there is an arc from r to s iff $a_{rs}\neq 0$. By a *walk* of lengh k in A we mean a formal product of some nonzero entries of A of the form $W=a_{i_0i_1}a_{i_1i_2}\cdots a_{i_{k-1}i_k}$ such a walk W is a called *path* if the indices $i_0,i_1,i_2,\cdots,i_k$ are distinct, except possibly $i_0=i_k$. Note that a walk W may be identified with the corresponding walk in the digraph $D(A)$. A *cycle* of length k in A is a nonzero product of the form $\gamma=a_{i_ki_1}a_{i_1i_2}\cdots a_{i_{k-1}i_k}$ where the indices $i_1,i_2,\cdots,i_k$ are distinct. For a walk of W in A, we define the *actual product* of W, denoted by $ap(W)$, to be the product of the entries in W.

We say that an $n\times n$ ray pattern A is *powerful* if for each positive integer k, the matrix A^k has no $\#$. For a powerful ray pattern A, consider the sequence $A=A^1,A^2,A^3,\cdots$. If this sequence has repeations, we say the ray pattern A is *periodic*. Let A^l be the first one that is repeated. Write $A^l=A^{l+p}$ with the minimal $p>0$. Then l is called the *base* of A, and p the *period* of A. Denote the base of A by $l(A)$, and the period of A by $p(A)$.

For a ray pattern $A=[a_{ij}]$, we define $|A|=[a'_{ij}]$ where $a'_{ij}=1$ if $a_{ij}\neq 0$ and $a'_{ij}=0$ if $a_{ij}=0$. Note that $|A|$ is a Boolean matrix. If $|D|$ is an identity matrix, we say D is a *diagonal ray pattern.* For ray patterns $A=[a_{ij}]$ and $B=[b_{ij}]$, we say B is *ray diagonally similar* to A if there exists a diagonal ray pattern D such that $A=DBD^*$. And if $a_{ij}=\delta_{ij}b_{ij}$ where δ_{ij} is 1 or 0 for all i,j, then we say A is a *subpattern* of B.

In this paper we consider the power of square ray patterns. Note that each powerful sign pattern A is periodic (See [2]). But for the ray pattern

$$A=e^i\begin{bmatrix}1&1\\1&1\end{bmatrix},$$

A is powerful but not periodic. In short, powerfulness does not guarantee that the ray pattern is periodic. The ray ω is *periodic* if there exists some positive integer m such that $\omega^m=1$. And if ω is periodic the smallest m satisfies the equation $\omega^m=1$ is called the *period* of ω and denote it by $p(\omega)$. In this example, we can say that A is not periodic since the ray e^i is not periodic.

Let A be an irreducible ray pattern with *index of imprimitivity* denoted by $k(A)=k$, where k is equal to the greatest common divisor of the lengths of the cycles in A. By adapting arguments on irreducible nonegative matrices, we see that A is permutationally similar to a ray pattern in block cyclic form, see [1]. For simplicity of notation, we may assume that A is already in block cyclic form:

$$A=\begin{bmatrix} O & A_{1,2} & & & \\ & O & A_{2,3} & & \\ & & \ddots & \ddots & \\ & & & O & A_{k-1,k} \\ A_{k,1} & & & & O \end{bmatrix} \qquad (1)$$

where the zero diagonal blocks are square, and the nonzero blocks have no zero row or zero column.

Note that if A is periodic, then A^k is also periodic. So each diagonal block of A^k, $A_{i,i+1}A_{i+1,i+2}\cdots A_{i-1,i}$ where the indices are modulo k, is periodic for all i. Now we represent some previous results which will be used in this paper.

Proposition 1. *(See Lemma 1.2 in [3]) The set of powerful ray patterns is closed under the following operations:*

(i) *multiplication by any ray;*
(ii) *transposition;*
(iii) *conjugate transposition(denoted by *);*
(iv) *diagonal similarity;*
(v) *permutational similarity;*
(vi) *direct sum;*
(vii) *taking subpatterns.*

Proposition 2. *(See Theorem 2.1 in [3]) Let A be an $n \times n$ entrywise nonzero ray pattern. Then A is powerful iff A is ray diagonally similar to $e^{i\theta}J$ for some $\theta \in R$.*

Proposition 3. *(See Theorem 3.5 in [3]) Every irreducible powerful ray pattern is a subpattern of an entrywise nonzero powerful ray pattern.*

By combining the above two propositions, we obtain the following theorem.

Theorem 4. *Suppose a ray pattern A is irreducible. Then A is powerful iff A is ray diagonally similar to $\omega|A|$ where ω is a ray.*

Proof. ($\Leftarrow$) It is trivial since $\omega|A|$ is powerful.
($\Rightarrow$) Since A is an irreducible powerful ray pattern, there exists an entrywise nonzero powerful ray pattern $\hat{A}$ such that A is subpattern of $\hat{A}$ by Proposition 3. And there exists a diagonal ray pattern D such that $D\hat{A}D^* = \omega J$ for some ray ω by Proposition 2. Since DAD^* is a subpattern of $D\hat{A}D^*$, $DAD^* = \omega K$ where K is a Boolean matrix. Let $A = (a_{ij})$ and $D = diag\{d_1, \cdots, d_n\}$. Then the (i, j)-th entry of DAD^* is $d_i a_{ij} \bar{d}_j$. So we have $|DAD^*| = |A| = |K|$. Therefore A is ray diagonally similar to $\omega|A|$. □

In this paper, we consider the relationship between the ray pattern A and $|A|$ in section 2. And we consider the base and the period of ray patterns in section 3.

2 Ray pattern A and $|A|$

Suppose an irreducible ray pattern A is powerful. By Theorem 4, we have a new block form for the irreducible ray pattern A. For simplicity of notation, we may assume that A is already of block cyclic form

$$A = \omega \begin{bmatrix} 0 & |A_{1,2}| & & & \\ & 0 & |A_{2,3}| & & \\ & & \ddots & \ddots & \\ & & & 0 & |A_{k-1,k}| \\ |A_{k,1}| & & & & 0 \end{bmatrix}. \qquad (2)$$

This means that there is ω such that A is ray diagonally similar to $\omega|A|$. Then how can we find such ω? First, consider the case when A is a cycle.

Lemma 5. *Suppose ray pattern A is in cyclic form*

$$A = \begin{bmatrix} 0 & \alpha_1 & & & \\ & 0 & \alpha_2 & & \\ & & \ddots & \ddots & \\ & & & 0 & \alpha_{k-1} \\ \alpha_k & & & & 0 \end{bmatrix},$$

where α_i is a ray for each i, $\alpha_1\alpha_2\cdots\alpha_k = \alpha \neq 0$ and each of off-diagonal entries is 0. Then A is ray diagonally similar to

$$\omega \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{bmatrix}$$

for each ω such that $\omega^k = \alpha$.

Proof. Suppose $\alpha = 1$. Let $arg(\alpha_i) = \theta_i$ for each i and $\theta \in R$. Consider d_i where $arg(d_1) = \theta$ and $arg(d_i) = \theta + \sum_{j=1}^{i-1}\theta_j$ for $i \geq 2$, and take $D = diag\{d_1, d_2, \cdots, d_k\}$. Then the arguments of $(i, i+1)$ entry of DAD^* is

$$arg(d_i\alpha_i d_{i+1}^*) = (\theta + \sum_{j=1}^{i-1}\theta_j) + \theta_i - (\theta + \sum_{j=1}^{i}\theta_j) = 0$$

where $2 \leq i \leq k-1$ and the argument is modulo 2π. And we can have $arg(d_1\alpha_1 d_2^*) = 0$ and $arg(d_k\alpha_k d_1^*) = 0$ since $arg(\alpha) = \sum_{j=1}^{k}\theta_j = 0$ modulo 2π. Therefore if $\alpha = 1$, A is ray diagonally similar to

$$DAD^* = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{bmatrix}.$$

In genetal case, let $B_\omega = \frac{1}{\omega}A$ for each ω such that $\omega^k = \alpha$. Since $B_\omega = \frac{1}{\omega}A$ is ray diagonally similar to

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{bmatrix},$$

we have A is ray diagonally similar to $\omega|A|$ for each ω such that $\omega^k = \alpha$. ∎

Consider the following matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Following the proof of the Lemma 5, we may find two diagonal ray patterns

$$D_1 = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$$

such that $D_1 A D_1^* = -i|A|$ and $D_2 A D_2^* = i|A|$ hold. Does there exist a third ray ω such that $\omega \neq i, -i$ and $DAD^* = \omega|A|$ for some diagonal ray pattern D?
Formally speaking, for an irreducible powerful ray pattern A, let

$$\Omega(A) = \{\omega | A \textit{ is ray diagonally similar to } \omega|A|\}.$$

What is the cardinality of $\Omega(A)$? In the following, we answer this question.

If a matrix A is of the form

$$\begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_s \end{bmatrix},$$

where each A_i is square for $1 \leq i \leq s$ and each of off-diagonal block submatrices is a zero matrix, then we denote A by $\bigoplus_{i=1}^{s} A_i$.

Lemma 6. *If an irreducible ray pattern A is of block cyclic form (2), then A is ray diagonally similar to $\alpha|A|$ for each α such that $\alpha^k = \omega^k$.*

Proof. By Lemma 5, for each α such that $\alpha^k = \omega^k$, there exists a diagonal ray pattern $D = diag\{d_1, \cdots, d_k\}$ such that

$$D \begin{bmatrix} 0 & \omega & & & \\ & 0 & \omega & & \\ & & \ddots & \ddots & \\ & & & 0 & \omega \\ \omega & & & & 0 \end{bmatrix} D^* = \alpha \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{bmatrix}$$

Let $E = \bigoplus_{i=1}^{k} d_i I_i$ where I_i is identity matrix for each i. Then the $(i, i+1)$ block of EAE^* is

$$d_i I_i (\omega |A_{i,i+1}|) \bar{d}_{i+1} I_{i+1} = \alpha |A_{i,i+1}|.$$

Therefore we have A is ray diagonally similar to $\alpha|A|$ for each α such that $\alpha^k = \omega^k$. □

Note that Lemma 6 says that $|\Omega(A)| \geq k(A)$ for irreducible powerful ray pattern A.

Lemma 7. *Suppose irreducible ray pattern A is powerful. If A is ray diagonally similar to $\omega|A|$ and $\omega'|A|$, then $\omega^{k(A)} = (\omega')^{k(A)}$.*

Proof. Let $k(A) = k$ and $L(A) = \{l_1, l_2, \cdots, l_m\}$ be the set of lengths of the cycles in A. First we show that if γ is a cycle in $D(A)$ whose length is s then $\omega^s = ap(\gamma)$. We can take a diagonal matrix $D = diag\{d_1, \cdots, d_n\}$ such that $DAD^* = \omega|A|$. And we have $DA^sD^* = \omega^s|A|^s$. Let i be a vertex on γ, then the (i, i) entry of DA^sD^* is $d_i ap(\gamma)\bar{d}_i$. Since $d_i\bar{d}_i = 1$, we have $\omega^s = ap(\gamma)$.

Since k is the greatest common divisor of $L(A)$, we can take integers $\alpha_1, \alpha_2, \cdots,$ α_m such that $\sum_{i=1}^{m} \alpha_i l_i = k$. Assume that $\omega,\ \omega' \in \Omega(A)$. Then we have

$$\omega^k = (\omega^{l_1})^{\alpha_1}(\omega^{l_2})^{\alpha_2} \cdots (\omega^{l_m})^{\alpha_m}.$$

And for each i, $(\omega^{l_i})^{\alpha_i} = (ap(\gamma_i))^{\alpha_i}$ where γ_i is a cycle of length l_i. So we have

$$\omega^k = (ap(\gamma_1))^{\alpha_1}(ap(\gamma_2))^{\alpha_2} \cdots (ap(\gamma_m))^{\alpha_m},$$

where γ_i is a cycle of length l_i. Same reasoning shows that

$$(\omega')^k = (ap(\gamma_1))^{\alpha_1}(ap(\gamma_2))^{\alpha_2} \cdots (ap(\gamma_m))^{\alpha_m}.$$

So $\omega^k = (\omega')^k$. □

Note that Lemma 7 means that $|\Omega(A)| \leq k(A)$.

By combining Lemma 6 and Lemma 7, we can obtain the following theorem.

Theorem 8. *Suppose irreducible ray pattern A is powerful. Then $|\Omega(A)| = k(A)$.*

In [2] , the authors introduce the concept of *cyclically nonnegative sign pattern.* We generalize this concept to ray patterns. A ray pattern A is *cyclically nonnegative* if the actual product of each cycle in A is 1.

Corollary 9. *Suppose an irreducible powerful ray pattern A is ray diagonally similar to $\omega|A|$ for some ray ω. A is cyclically nonnegative iff $\omega^{k(A)} = 1$.*

Proof. Let $k(A) = k$, and let $L(A)$ and $\alpha_1, \alpha_2, \cdots, \alpha_m$ be the same as in the proof of Theorem 7. Suppose $\omega^k = 1$. Since $k|l_i$ for each i, $\omega^{l_i} = 1$. Next, suppose A is cyclically nonnegative. Same reasoning as in the proof of Theorem 7 shows that

$$\omega^k = (ap(\gamma_1))^{\alpha_1}(ap(\gamma_2))^{\alpha_2} \cdots (ap(\gamma_m))^{\alpha_m}.$$

By assumption, $ap(\gamma_i) = 1$ for each i. Thus $\omega^k = 1$. This completes the proof □

In the proof of Lemma 7, we actually prove the following proposition.

Proposition 10. *Suppose irreducible ray pattern A is powerful. If A is ray diagonally similar to $\omega|A|$, then $\omega^s = ap(\gamma)$ for each cycle γ in $D(A)$ whose length is s.*

Now we can answer the question which was given in the middle of this section. Let A be a ray pattern

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Note that $|\Omega(A)| = k(A) = 2$ by Theorem 8. Furthermore, since $D(A)$ has only one cycle whose length is 2 with actual product -1, thus we can find $\Omega(A) = \{i, -i\}$ without considering the diagonal ray pattern D by Theorem 8 and Proposition 10.

3 The bases and the periods of ray patterns

For brevity, when we say a ray pattern A is periodic, we assume that A is powerful. In this section, we suggest another characterizations of periodic ray patterns.

Proposition 11. *(See Lemma 1.2 in [2]) Suppose powerful ray pattern A is periodic. Then for positive integers m, k, $A^m = A^{m+k}$ iff $m \geq l(A)$ and $p(A)|k$.*

Note that Proposition 11 in [2] considered the case that A is a powerful sign pattern. But we can obtain the Proposition 11 by a slight modification of the statement in origin. Now, the following result is easily obtained from Theorem 4.

Theorem 12. *If a periodic irreducible ray pattern A is ray diagonally similar to $\omega|A|$, then we have $l(A) = l(|A|)$ and $p(A) = lcm\{p(\omega), p(|A|)\}$. In particular, if $k(A) = k$ then we have $p(A) = p(\omega^k)k$.*

Proof. We may assume $A = \omega|A|$ since the base and the period are invariant under ray diagonal similarity. Let $lcm\{p(\omega), p(|A|)\} = p$. Then

$$\begin{aligned} A^{l(A)+p(A)} &= A^{l(A)}, \\ \omega^{l(A)+p(A)}|A|^{l(A)+p(A)} &= \omega^{l(A)}|A|^{l(A)}, \\ \omega^{p(A)}|A|^{l(A)+p(A)} &= |A|^{l(A)}. \end{aligned}$$

Since each nonzero entry of $|A|$ is 1, $\omega^{p(A)}$ must be 1. Hence $p(\omega)|p(A)$. So from the last equality, we have $|A|^{l(A)+p(A)} = |A|^{l(A)}$. Thus $l(A) \geq l(|A|)$ and $p(|A|)|p(A)$ by Proposition 11. Therefore $l(A) \geq l(|A|)$ and $p|p(A)$. Also we can obtain the following equations.

$$
\begin{aligned}
|A|^{l(|A|)+p(|A|)} &= |A|^{l(|A|)},\\
|A|^{l(|A|)+p} &= |A|^{l(|A|)},\\
\omega^{l(|A|)+p}|A|^{l(|A|)+p} &= \omega^{l(|A|)+p}|A|^{l(|A|)},\\
\omega^{l(|A|)+p}|A|^{l(|A|)+p} &= \omega^{l(|A|)}|A|^{l(|A|)},\\
A^{l(|A|)+p} &= A^{l(|A|)}.
\end{aligned}
$$

So we have $l(|A|) \geq l(A)$ and $p(A)|p$. Therefore we have $l(A) = l(|A|)$ and $p(A) = p = lcm\{p(\omega), p(|A|)\}$.

Let $p = lcm\{p(\omega), k\}$ and $\alpha = p(\omega^k)$ and remind $p(|A|) = k$(See [1]). We have $(\omega^k)^\alpha = \omega^{\alpha k} = 1$. Since $p(\omega)|\alpha k$, we have $p|\alpha k$. Conversely, we can have $(\omega^p)^k = 1$. Since $\alpha|p$ and $k|p$, we have $\alpha k|p$. So $\alpha k = p$. This completes the proof. □

Let A be an irreducible powerful sign pattern. Then any actual product of a cycle in A is 1 or -1. Suppose A is ray diagonally similar to $\omega|A|$. Then by Corollary 9, $\omega^{k(A)} = 1$ iff A is cyclically nonnegative. Otherwise $\omega^{k(A)} = -1$. So from Theorem 12, we have the following result.

$$
p(A) = \begin{cases} k & \text{if A is cyclically nonnegative} \\ 2k & \text{if A has a negative cycle} \end{cases}
$$

and

$$
l(A) = l(|A|).
$$

So we can consider Theorem 12 as a generalization of Theorme 4.3 in [2].

Now we characterize a periodic irreducible ray pattern whose period is p.

Theorem 13. *Suppose A is an irreducible ray pattern with $k(A) = k$. A is periodic with $p(A) = p$ iff k divides p, and A is ray diagonally similar to $\omega|A|$ where $p(\omega^k) = p/k$.*

Proof. ($\Rightarrow$) Note that A is ray diagonally similar to $\omega|A|$ for some ω by Theorem 4 and $p(A) = p(\omega^k)k$ by Theorem 12. Therefore A is ray diagonally similar to $\omega|A|$ where $p = p(\omega^k)k$.
($\Leftarrow$) Note that A is periodic since ω is periodic, and $p(A) = p(\omega|A|) = p(\omega^k)k = p$ since $k(|A|) = k$. □

By the above Theorem 13, we can obtain the following corollary about pattern p-potents, which was already presented in [4].

Corollary 14. *Suppose A is an irreducible ray pattern in block cyclic form (1) with $k(A) = k$. A is pattern p-potent for some positive integer p iff k divides p, and A*

is ray diagonally similar to

$$\omega \begin{bmatrix} 0 & J_1 & & & & \\ & 0 & J_2 & & & \\ & & \ddots & \ddots & & \\ & & & & 0 & J_{k-1} \\ J_k & & & & & 0 \end{bmatrix},$$

where $p(\omega^k) = p/k$ *and* J_i *is an all ones matrix that is the same size as the corresponding block* $A_{i,i+1}$.

Proof. Note that A is ray diagonally similar to $\omega|A|$ where $p(\omega^k) = p/k$ by Theorem 13. Since $l(A) = l(|A|) = 1$, we have

$$|A| = \begin{bmatrix} 0 & J_1 & & & & \\ & 0 & J_2 & & & \\ & & \ddots & \ddots & & \\ & & & & 0 & J_{k-1} \\ J_k & & & & & 0 \end{bmatrix},$$

where J_i is an all ones matrix that is the same size as the corresponding block $A_{i,i+1}$. This completes the proof. □

4 Closing remark

In this paper, we study irreducible powerful ray patterns using Theorem 4. One of key observation of our paper is that if a powerful ray pattern A is irreducible, then A is ray diagnally similar to $\omega|A|$ for some ray ω. Now consider the following powerful ray pattern set

$$S = \{A|A \textit{ is ray diagnally similar to } \omega|A| \textit{ for some ray } \omega\}.$$

Note every powerful irreducible ray patterns is an element of S. Then how can we characterize the elements in S, and what are the bases and periods of the elements in S?

For examples, consider the following reducible powerful ray patterns

$$A = \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then A is ray diagnally similar to $i|A|$ and $-i|A|$. Thus A is in S, but B is not. How can we determine whether a powerful ray pattern is in S or not?

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