LINEARIZING ℓ-ORDER GENERALIZED SYSTEMS. CONTROLLABILITY OF LINEARIZED SYSTEMS

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Abstract. Given a $(\ell+2)$-ple of matrices $(E, A_{\ell-1}, \ldots, A_0, B)$ representing ℓ-order generalized time-invariant dynamical systems, $Ex^{(\ell)} = A_{\ell-1}x^{(\ell-1)} + \ldots + A_0x^{(0)} + Bu$ ($x^{(i)}$ denotes the i-th derivative of x), we analyze conditions for which there exists a control $u_1 = u + F_\ell x^{(\ell)} - \ldots - F_0x^{(0)}$ the new system can be linearized, and the linearized system has a stable solution.

Key words. High-order dynamical systems, linearization, feedback, controllability.

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1. Introduction. We consider the space $\mathcal{M}$ of $(\ell+2)$-ples of matrices $(E, A_{(\ell-1)}, \ldots, A_0, B)$ where $E, A_{(\ell-1)}, \ldots, A_0 \in M_n(\mathbb{C})$, and $B \in M_{n\times m}(\mathbb{C})$ corresponding to a ℓ-order generalized time-invariant linear systems

$$Ex^{(\ell)} = A_{\ell-1}x^{(\ell-1)} + \ldots + A_0x^{(0)} + Bu, \tag{1.1}$$

($x^{(i)}$ denotes the i-th derivative).

When $E = I_n$ it is called *standard ℓ-order linear system*

$$x^{\ell} = A_{\ell-1}x^{(\ell-1)} + \ldots + A_0x^{(0)} + Bu \tag{1.2}$$

and we write simply as a $(\ell+1)$-ple of matrices $(A_{\ell-1}, \ldots, A_0, B)$.

It is well known that, standard ℓ-order linear systems may be linearized See [5] for example, in the sense that the system can be transformed to a linear system in the form $X^{(1)} = \mathbf{A}X + \mathbf{B}u$.

We say that a ℓ-order generalized system is *standardizable* if the matrix E is invertible because in this case, by pre-multiplication by E^{-1}, the equation of the system (1.1), is transformed to a standard one and consequently, it can be linearized.

We ask if it is possible by means of the introduction of a *ℓ-order derivative feedback* $u = u_1 - F_\ell x^{(\ell)} + \ldots + F_0x^{(0)}$ on the generalized time-invariant equation (1.1), to transform the system to another $(E + BF_\ell)x^{(\ell)} = (A_{\ell-1} + BF_{\ell-1})x^{(\ell-1)} + \ldots + (A_0 + BF_0)x^{(0)} + Bu_1$ that it is standardizable and the linearized system has a stable solution. In this case we say that the system (1.1) may be "standardized" by a "ℓ-order derivative feedback" or that the system (1.1) is "standardizable" by a ℓ-order derivative feedback.

As the case of standard ℓ-order systems, the standardized system may be linearized, and in this we can analyze the controllability of the linear system obtained. In this paper we obtain conditions from the initial ℓ-order systems to ensure the controllability of the linearized standardized ℓ-order system.

The study of generalized linear systems is being of a great deal of interest in recent years. Derivative feedback is used by Rath [6] in order to regularize generalized systems with variable coefficients. Standardizable first order generalized linear systems by means a derivative feedback has been recently studied [3], [4]. Concerning second order generalized systems an algorithm to compute the transfer function $(s^2E - sA_1 - A_0)B$ has been obtained by G. Antoniou [1].

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2. Linearizing standard ℓ-order linear systems. Given a ℓ-order standard linear system $x^{(\ell)} = A_{\ell-1}x^{(\ell-1)}+\ldots+A_0x^{(0)}+Bu$ or simply write $(A_{\ell-1},\ldots,A_0,B)$, it is well known that it can be linearized in the following manner. Calling $X = \begin{pmatrix} x^{(0)} \\ x^{(1)} \\ \vdots \\ x^{(\ell-1)} \end{pmatrix}$, we have the following linear system

$$X^{(1)} = \mathbf{A}X + \mathbf{B}u \tag{2.1}$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & I_n & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \ldots & I_n \\ A_0 & A_1 & \ldots & A_{\ell-1} \end{pmatrix} \in M_{\ell n}(\mathbb{C}), \quad \mathbf{B} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ B \end{pmatrix} \in M_{\ell n\times m}(\mathbb{C}) \tag{2.2}$$

We have also that, if $\begin{pmatrix} x^{(0)}(t) \\ \vdots \\ x_0^{(\ell-1)}(t) \end{pmatrix}$ is a solution of the linear system (2.1), then $x^{(0)}(t)$ is a solution of the ℓ-order equation (1.2). And conversely, if $x_0^{(0)}(t)$ is a solution of the ℓ-order equation (2.1), then $\begin{pmatrix} x_0^{(0)}(t) \\ \vdots \\ x_0^{(\ell-1)}(t) \end{pmatrix}$ is a solution of the linear system (2.1)

If we can consider feedback equivalent linear system in the form (2.2) we need to restrict the feedback group to the subgroup formed by $(\ell+2)$-ples of matrices $(P,Q,F_0,\ldots,F_{\ell-1})$ $P\in Gl(n;\mathbb{C})$, $Q\in Gl(m;\mathbb{C})$, and $F_i\in M_{m\times n}(\mathbb{C})$ acting over the space of this kind of systems in the following manner

DEFINITION 2.1. *Two systems $(A^i_{\ell-1},\ldots,A^i_0,B^i)$, $i=1,2$, are equivalent, if and only if, the exist matrices $P\in Gl(n;\mathbb{C})$, $Q\in Gl(m;\mathbb{C})$, and $F_i\in M_{m\times n}(\mathbb{C})$ such that*

$$\begin{pmatrix} 0 & I_n & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \ldots & I_n & 0 \\ A_0^2 & A_1^2 & \ldots & A^2_{\ell-1} & B^2 \end{pmatrix} = \tag{2.3}$$
$$\begin{pmatrix} P^{-1} & & 0 \\ & \ddots & \\ 0 & & P^{-1} \end{pmatrix} \begin{pmatrix} 0 & I_n & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & I_n & 0 \\ A_0^1 & A_1^1 & \ldots & A^1_{\ell-1} & B^1 \end{pmatrix} \begin{pmatrix} P & & 0 & 0 \\ & \ddots & & \\ 0 & & P & 0 \\ F_0 & \ldots & F_{\ell-1} & Q \end{pmatrix}$$

That is to say, the transformations permitted over ℓ-order standard systems are basis change in the state space $x=Px_1$, in the input space $u=Qu_1$ and i-order derivative feedback $(i=0,\ldots,\ell-1)$ $u=u_1+F_0x^{(0)}+\ldots+F_{\ell-1}x^{(\ell-1)}$.

With this definition we ensure that equivalent systems to a linearized system are linearized systems.

From about definition, we have the following proposition.

PROPOSITION 2.2. *Let $x^{(\ell)} = A^i_{\ell-1}x^{(\ell-1)}+\ldots+A^i_0x^{(0)}+B^i$ $i=1,2$ two equivalent ℓ-order standard linear systems. Then the linearized systems are feedback equivalent.*

Notice that if $(A^i_{\ell-1},\ldots,A^i_0,B^i)$ $i=1,2$ are two equivalent systems, then each one of the pairs of matrices $(A^1_{\ell-1},B^1)$, …, (A^1_0,B^1) is feedback equivalent to the pair $(A^2_{\ell-1},B^2)$, …, (A^2_0,B^2) respectively. Then, and if necessary we can take systems $(A_{\ell-1},\ldots,A_0,B)$ where one of the pairs $(A_{\ell-1},B)$, … or (A_0,B) is in a canonical reduced form (Kronecker reduced form, for example).

3. Controllability. We can apply controllability results about linear systems and we obtain the following proposition.

PROPOSITION 3.1. *Let $x^{(\ell)} = A_{\ell-1}x^{(\ell-1)} + \ldots + A_0x^{(0)} + Bu$ be a ℓ-order linear system. The linearized systems $\dot{X} = \mathbf{A}X + \mathbf{B}u$ is controllable, if and only if,*

$$\text{rank} \begin{pmatrix} s^\ell I_n - s^{\ell-1}A_{\ell-1} - \ldots - sA_1 - A_0 & B \end{pmatrix} = n, \qquad \forall s \in \mathbb{C} \tag{3.1}$$

Proof. It is well known that, the system $\dot{X} = \mathbf{A}X + \mathbf{B}u$ is controllable, if and only if,

$$\text{rank} \begin{pmatrix} sI_{\ell n} - \mathbf{A} & \mathbf{B} \end{pmatrix} = \ell n, \qquad \forall s \in \mathbb{C} \tag{3.2}$$

making row and column elementary transformations to the matrix $\begin{pmatrix} sI_{\ell n} - \mathbf{A} & \mathbf{B} \end{pmatrix}$ we have

$$\begin{array}{l} \text{rank} \begin{pmatrix} sI_n & -I_n & 0 & \ldots & 0 & 0 & 0 \\ 0 & sI_n & -I_n & \ldots & 0 & 0 & 0 \\ & \ddots & \ddots & & & & \\ 0 & 0 & 0 & \ldots & sI_n & -I_n & 0 \\ -A_0 & -A_1 & -A_2 & \ldots & -A_{\ell-2} & sI_n - A_{\ell-1} & B \end{pmatrix} = \\ \text{rank} \begin{pmatrix} 0 & I_n & 0 & \ldots & 0 & 0 \\ 0 & 0 & I_n & \ldots & 0 & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ 0 & 0 & 0 & \ldots & I_n & 0 \\ s^\ell I_n - s^{\ell-1}A_{\ell-1} - \ldots - sA_1 - A_0 & 0 & 0 & \ldots & 0 & B \end{pmatrix} \end{array} \tag{3.3}$$

□

It is not difficult to prove that for some particular cases we have the following results for second-order linear systems

PROPOSITION 3.2.

1.- Case $A_1 = 0$. The linearized system is controllable if and only if the pair of matrices (A_0, B) is controllable.

2.- Case $A_0 = 0$. It is not difficult to prove that the linearized system is controllable if and only if $n \geq m$ and the matrix B has full rank

Let $x^{(\ell)} = A_{\ell-1}x^{(\ell-1)} + \ldots + A_0x^{(0)} + Bu$ be a ℓ-order system where the linearized system $\dot{X} = \mathbf{A}X + \mathbf{B}u$ being controllable, then it is well known that, there exists a control $u_1 = u - F_{\ell-1}x^{(\ell-1)} + \ldots + F_0x^{(0)}$, such that the linear equation $\dot{X} = \mathbf{A}X + \mathbf{B}u_1$ has a stable solution $\begin{pmatrix} x_0^{(0)}(t) \\ \vdots \\ x^{(\ell-1)}{}_0(t) \end{pmatrix}$. Taking into account that $x_0^{(0)}(t)$ is a solution of the ℓ-order equation, we have that the ℓ-order equation has a stable solution.

4. Standardizable systems. Now, we are interested in the kind of $(\ell + 2)$-ples $(E, A_{\ell-1}, \ldots, A_0, B)$ which there exist a matrix F in such a way $E + BF$ being invertible, that induce to consider the following equivalence relation generalizing 2.1

DEFINITION 4.1. *Two $(\ell+2)$-ples $(E^i, A^i_{\ell-1}, \ldots, A^i_0, B^i)$, $i = 1, 2$, are equivalent if and only if there exist matrices $P \in Gl(n; \mathbb{C})$, $Q \in Gl(m; \mathbb{C})$ and $F_i \in M_{m\times n}(\mathbb{C})$ $i = 0, \ldots, \ell$, such that*

$$\begin{aligned} E^2 &= P^{-1}E^1P + P^{-1}BF_\ell, \\ A^2_{\ell-1} &= P^{-1}A^1_1P + P^{-1}B^1F_{\ell-1}, \\ &\vdots \\ A^2_0 &= P^{-1}A^1_2P + P^{-1}B^1F_0, \\ B^2 &= P^{-1}B^1Q. \end{aligned} \tag{4.1}$$

or in a matrix form

$$\begin{pmatrix} 0 & I_n & \dots & 0 & 0 & 0 \\ \\ 0 & 0 & \dots & I_n & 0 & 0 \\ A_0^2 & A_1^2 & \dots & A_{\ell-1}^2 & E^2 & B^2 \end{pmatrix} = \tag{4.2}$$
$$\begin{pmatrix} P^{-1} & & 0 \\ & \ddots & \\ 0 & & P^{-1} \end{pmatrix} \begin{pmatrix} 0 & I_n & \dots & 0 & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & \dots & I_n & 0 & 0 \\ A_0^1 & A_1^1 & \dots & A_{\ell-1}^1 & E^1 & B^1 \end{pmatrix} \begin{pmatrix} P & \dots & 0 & 0 \\ & \ddots & & \\ 0 & \dots & P & 0 \\ F_0 & \dots & F_\ell & Q \end{pmatrix}$$

Like as for $(\ell+1)$-ples we have that if $(E^i, A^i_{\ell-1}, \dots, A^i_0, B^i)$ $i = 1, 2$ are two equivalent generalized linear systems, then each one of the pairs of matrices (E^1, B^1), $(A^1_{\ell-1}, B^1)$, ... and (A^1_0, B^1) are feedback equivalent to the pairs (E^2, B^2), $(A^2_{\ell-1}, B^2)$, ... and (A^0_2, B^2) respectively. Then, and if necessary we can take systems $(E, A_{\ell-1}, \dots, A_0, B)$ where one of the pairs (E, B), $(A_{\ell-1}, B)$, ... or (A_0, B) is in a canonical reduced form (Kronecker reduced form, for example).

LEMMA 4.2. *Let $(E^1, A^1_{\ell-1}, \dots, A^1_0, B^1)$ be a $(\ell+2)$-ple with E^1 invertible. Then, for any $(\ell+2)$-ple $(E^2, A^2_{\ell-1}, \dots, A^2_0, B^2)$ equivalent to it, there exists a matrix F such that $E^2 + B^2F$ is invertible.*

Proof. The equivalence relation ensures that $E^2 = P^{-1}E^1P + P^{-1}B^1F_\ell$ and $B^2 = P^{-1}B^1Q$. Then $E^2 - B^2Q^{-1}F_\ell = P^{-1}E^1P$ is invertible. So, taking $F = -Q^{-1}F_\ell$ the matrix $E^2 + B^2F$ is invertible. □

LEMMA 4.3. *Let $(E^1, A^1_{\ell-1}, \dots, A^1_0, B^1)$ be a $(\ell+2)$-ple such that there exists a matrix F_ℓ with $E^1 + B^1F_\ell$ invertible. Then, for any $(\ell+2)$-ple $(E^2, A^2_{\ell-1}, \dots, A^2_0, B^2)$ equivalent to it, there exists a matrix F such that $E^2 + B^2F$ is invertible.*

Proof. Obviously the $\ell+2$-ple $(E^1 + BF_\ell, A^1_{\ell-1}, \dots, A^1_0, B^1)$ is equivalent to $E^1, A^1_{\ell-1}, \dots, A^1_0, B^1$, so it is equivalent to $E^2, A^2_{\ell-1}, \dots, A^2_0, B^2$. Now it suffices to apply the previous lemma. □

THEOREM 4.4. *A ℓ-order generalized system $Ex^{(\ell)} = A_{\ell-1}x^{(\ell-1)} + \dots + A_0x^{(0)} + Bu$ may be standardized by a ℓ-order derivative feedback if, and only if, the matrix $\begin{pmatrix} E & B \end{pmatrix}$ has full rank.*

Proof. Lemma before permit us to consider an equivalent ℓ-order generalized system where the pair (E, B) is in its Kronecker reduced form. □

Example: Let $(E, A_{\ell-1}, \dots, A_2, B)$ be a $(\ell+2)$-ple with $E = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The pair (E, B) is in such a way that all its eigenvalues are non-zero. Then there exists F_ℓ (we can take $F_\ell = \begin{pmatrix} 1 & 0 \end{pmatrix}$) such that $E + BF_\ell$ is an invertible matrix and the second-order generalized system can be standardized.

Remark We observe that to ensure standardization it suffices to consider ℓ-derivative feedback in the form $u_1 = u + F_\ell x^\ell$. But it is not sufficed to ensure stable solution, as we can see in the following example

Example: Let $Ex^{(1)} = Ax + Bu$ with $E = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Taking $F_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}$ we have that $E + BF_1 = I$. Then, the standardized system is $x^{(1)} = Ax + Bu$. The eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 3$, so the matrix A is not stable, but if we consider the control $u_2 = u + F_1x^{(1)} - F_0x$ with $F_0 = \begin{pmatrix} -\frac{25}{6} & -\frac{40}{6} \end{pmatrix}$, the eigenvalues of the system $x^{(1)} = (E+BF_1)^{-1}(A+BF_0)x + (E+BF_1)^{-1}Bu_2$ are $\frac{1}{2}$, $\frac{1}{3}$ and the matrix $(E+BF_1)^{-1}(A+BF_0)$ is stable.

Notice that the linearized standardized system is controllable.

5. Controllability of standardized ℓ-systems. The last example in §4, induce us to study the controllability of standardized ℓ-order systems, and we ask if it is possible to know something about controllability, directly from the ℓ-order generalized system. We have the following proposition.

PROPOSITION 5.1. *Let $Ex^{(\ell)} = A_{\ell-1}x^{(\ell-1)} + \ldots + A_1x^{(1)} + A_0x + Bu$ a standardizable ℓ-order system. The standardized system is controllable if and only if*

$$\text{(5.1)} \qquad \textit{rank} \begin{pmatrix} s^\ell E - s^{\ell-1}A_{\ell-1} - \ldots - sA_1 - A_0 & B \end{pmatrix} = n, \qquad \forall s \in \mathbb{C}.$$

Proof. The standardized system

$$\text{(5.2)} \quad x^\ell = (E+BF_\ell)^{-1}A_{\ell-1}x^{(\ell-1)} + \ldots + (E+BF_\ell)^{-1}A_1x + (E+BF_\ell)^{-1}A_0 + (E+BF_\ell)^{-1}Bu$$

is controllable if and only if

$$\text{(5.3)} \ \text{rank} \begin{pmatrix} s^\ell I_n - s^{\ell-1}(E+BF_\ell)^{-1}A_\ell - \ldots - (E+BF_\ell)^{-1}A_0 & (E+BF_\ell)^{-1}B \end{pmatrix} = n, \quad \forall s \in \mathbb{C}$$

But

$$\text{(5.4)} \quad \begin{aligned} &\begin{pmatrix} s^\ell I_n - s^{\ell-1}(E+BF_\ell)^{-1}A_\ell - \ldots - (E+BF_\ell)^{-1}A_0 & (E+BF_\ell)^{-1}B \end{pmatrix} = \\ &(E+BF_\ell)^{-1} \begin{pmatrix} s^\ell E - s^{\ell-1}A_{\ell-1} - \ldots - sA_1 - A_0 & B \end{pmatrix} \begin{pmatrix} I_n & 0 \\ s^\ell F_\ell & I_m \end{pmatrix} \end{aligned}$$

□

As a consequence we obtain the following theorem.

THEOREM 5.2. *Let $Ex^\ell = A_{\ell-1}x^{(\ell-1)} + \ldots + A_0x^{(0)} + Bu$ be a ℓ-order linear equation with*

$$\text{(5.5)} \qquad \begin{aligned} &\textit{rank} \begin{pmatrix} E & B \end{pmatrix} = n \\ &\textit{rank} \begin{pmatrix} s^\ell E - s^{\ell-1}A_{\ell-1} - \ldots - sA_1 - A_0 & B \end{pmatrix} = n \ \forall s \in \mathbb{C}. \end{aligned}$$

Then, there exists a control $u_1 = u + F_\ell x^{(\ell)} - F_{\ell-1}x^{(\ell-1)} - \ldots - F_0x^0$, such that the equation $Ex^{(\ell)} = A_{\ell-1}x^{(\ell-1)} + \ldots + A_0x^{(0)} + Bu_1$ has a stable solution.

Remark: The set $\mathcal{S} = \{s_0 \in \mathbb{C}; \text{rank} \begin{pmatrix} s_0^\ell E - s_0^{\ell-1}A_{\ell-1} - \ldots - s_0A_1 - A_0 & B \end{pmatrix} < n\}$ is invariant under equivalence relation considered.

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