On the Schur Complement of Diagonally Dominant Matrices *

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1 Introduction

In 1979, Carlson and Markham proved that the Schur complements of strictly diagonally dominant matrices are strictly diagonally dominant [1]. Their proof relies virtually on the Crabtree-Haynsworth quotient formula and mathematical induction. The importance of the result is two-fold. In numerical analysis, the convergence of the Gauss-Seidel iterations is guaranteed for a strictly diagonally dominant matrix (see, e.g., [4, p. 58] or [3, p. 508]). In combinatorial matrix theory, the diagonally dominant matrices, along with other type of matrices such as positive semidefinite matrices, M-matrices, inverse M-matrices, are to join the classes that possess the closure properties under the Schur complement.

The purpose of this note is to present a conceptual proof, without using the quotient formula, to the existing result that the Schur complement of a diagonally dominant matrix is diagonally dominant.

Let A be an $n \times n$ complex matrix partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} is $k \times k$ and nonsingular, $1 \le k < n$. The *Schur complement* of A_{11} in A

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is defined to be

$$A/A_{11} \stackrel{\text{def}}{=} A_{22} - A_{21}A_{11}^{-1}A_{12}.$$

The definition of Schur complement may be generalized to singular A_{11} by replacing the inverse A_{11}^{-1} with a generalized inverse of A_{11} and to any (principal) submatrix of A via permutations. We shall consider the Schur complement of a $k \times k$ nonsingular principal submatrix that is located in the upper left corner.

It has been evident that the matrix identity $(A/A_{11})^{-1} = (A^{-1})_{22}$ (see, e.g., [5, p. 184]), relating submatrices and Schur complements, plays a key role in proving the closure properties of the classes such as inverse M-matrices under the Schur complement. This approach does not work for diagonally dominant matrices.

For an $n \times n$ complex matrix $A = (a_{ij})$, we say that A is (row) *diagonally dominant* if for all $i = 1, 2, \ldots, n$

$$a_{ii} \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|.$$

A is further called *strictly diagonally dominant* if all the strict inequalities hold. It is well-known that strictly diagonally dominant matrices are nonsingular. Obviously, the principal submatrices of strictly diagonally dominant matrices are strictly diagonally dominant and thus nonsingular.

The *comparison matrix* $\mu(A) = (c_{ij})$ of a given matrix $A = (a_{ij})$ is defined by

$$c_{ij} = \begin{cases} |a_{ij}| & \text{if } i = j \\ -|a_{ij}| & \text{if } i \neq j \end{cases}$$

We say that matrix A is an *H-matrix* if $\mu(A)$ is an M-matrix. Recall that a square matrix M is an *M-matrix* if it can be written in the form $M = \alpha I - P$, where P is a nonnegative matrix and $\alpha > \rho(M)$, the spectral radius of M.

We shall adopt the following notations:

$\mathcal{C}^{n\times n}$ for the set of all $n \times n$ complex matrices;

$\mathcal{D}_n$ for the set of all $n \times n$ diagonally dominant matrices;

$\mathcal{SD}_n$ for the set of all $n \times n$ strictly diagonally dominant matrices;

$\mathcal{M}_n$ for the set of all $n \times n$ M-matrices;

$\mathcal{H}_n$ for the set of all $n \times n$ H-matrices.

2 Theorem and the Proof

For any matrix $A = (a_{ij})$, we denote $|A| = (|a_{ij}|)$. If the entries of matrix A are all nonnegative, then we write $A \geq 0$. For real matrices A and B of the same size, if $A - B$ is a nonnegative matrix, we write $A \geq B$.

We begin with two well-known results (see, e.g., [2, p. 117 or p. 131] and [2, p. 114, Theorem 2.5.3.13], respectively).

Lemma 1. *Let $A \in \mathcal{C}^{n\times n}$ and $B \in \mathcal{M}_n$. If $\mu(A) \geq B$, then $A \in \mathcal{H}_n$ and*

$$B^{-1} \geq |A^{-1}| \geq 0.$$

Lemma 2. *Let $A \in \mathcal{SD}_n$. Then $\mu(A) \in \mathcal{M}_n$; that is, $A \in \mathcal{H}_n$.*

It follows immediately from Lemma 1 that

$$A \in \mathcal{H}_n \quad \Rightarrow \quad [\mu(A)]^{-1} \geq |A^{-1}|. \tag{1}$$

Now we state and prove the main result.

Theorem 3. *Let $A \in \mathcal{SD}_n$ and A_{11} be a $k \times k$ nonsingular leading principal submatrix of A. Then $A/A_{11} \in \mathcal{SD}_{n-k}$. In other words, if A is strictly diagonally dominant, then so is a Schur complement in A.*

Proof. Denote $j_i = k+i$, $i = 1, 2, \ldots, n-k$, and let the (s,t)-entry of A/A_{11} be a'_{st}. By definition and upon computation, we have

$$\begin{aligned}
&|a'_{tt}| - \sum_{u\neq t}^{n-k} |a'_{tu}| \\
&= \left| a_{j_tj_t} - (a_{j_t1}, \cdots, a_{j_tk}) A_{11}^{-1} \begin{pmatrix} a_{1j_t} \\ \vdots \\ a_{kj_t} \end{pmatrix} \right| \\
&\qquad - \sum_{u\neq t}^{n-k} \left| a_{j_tj_u} - (a_{j_t1}, \cdots, a_{j_tk}) A_{11}^{-1} \begin{pmatrix} a_{1j_u} \\ \vdots \\ a_{kj_u} \end{pmatrix} \right| \\
&\geq |a_{j_tj_t}| - \sum_{u\neq t}^{n-k} |a_{j_tj_u}| - \sum_{u=1}^{n-k} \left| (a_{j_t1}, \cdots, a_{j_tk}) A_{11}^{-1} \begin{pmatrix} a_{1j_u} \\ \vdots \\ a_{kj_u} \end{pmatrix} \right| \\
&\geq |a_{j_tj_t}| - \sum_{u\neq t}^{n-k} |a_{j_tj_u}| - \sum_{u=1}^{n-k} (|a_{j_t1}|, \cdots, |a_{j_tk}|) \, |A_{11}^{-1}| \begin{pmatrix} |a_{1j_u}| \\ \vdots \\ |a_{kj_u}| \end{pmatrix} \\
&\geq |a_{j_tj_t}| - \sum_{u\neq t}^{n-k} |a_{j_tj_u}| - \sum_{u=1}^{n-k} (|a_{j_t1}|, \cdots, |a_{j_tk}|) [\mu(A_{11})]^{-1} \begin{pmatrix} |a_{1j_u}| \\ \vdots \\ |a_{kj_u}| \end{pmatrix} \\
&\quad \text{(by (1))}
\end{aligned}$$

$$= \frac{1}{\det \mu(A_{11})} \det \begin{pmatrix} |a_{j_t j_t}| - \sum_{u \neq t}^{n-k} |a_{j_t j_u}| & -|a_{j_t 1}| & \cdots & -|a_{j_t k}| \\ - \sum_{u=1}^{n-k} |a_{1 j_u}| & & & \\ \vdots & & \mu(A_{11}) & \\ - \sum_{u=1}^{n-k} |a_{k j_u}| & & & \end{pmatrix}$$
$$\stackrel{\text{def}}{=} \frac{1}{\det \mu(A_{11})} \det B.$$

We now show that $B = (b_{st})$ is in $\mathcal{SD}_n$ (or an M-matrix). Since $A \in \mathcal{SD}_n$,

$$b_{11} = |a_{j_t j_t}| - \sum_{u \neq t}^{n-k} |a_{j_t j_u}| > 0,$$

$$b_{v+1,v+1} = |a_{vv}| > 0, \quad v = 1, \ldots, k,$$

$$b_{11} = |a_{j_t j_t}| - \sum_{u \neq t}^{n-k} |a_{j_t j_u}| > \sum_{v=1}^{k} |a_{j_t v}| = \sum_{v \neq 1}^{k+1} |b_{1v}|,$$

$$b_{v+1,v+1} = |a_{vv}| > \sum_{w \neq v}^{k} |a_{vw}| + \sum_{u=1}^{n-k} |a_{v j_u}| = \sum_{w \neq v+1}^{k+1} |b_{v+1,w}|, \quad v = 1, \ldots, k.$$

It follows that $B(= \mu(B))$ is strongly diagonally dominant and thus is an M-matrix of size $k+1$ (by Lemma 2). Therefore $\det B > 0$, along with $\det \mu(A_{11}) > 0$, yields $|a'_{tt}| - \sum_{u \neq t}^{n-k} |a'_{tu}| > 0$ for all $t = 1, 2, \ldots, n-k$. We conclude that A/A_{11} is strongly diagonally dominant. □

Corollary 4. *Let A be an $n \times n$ matrix with nonsingular submatrix A_{11}. Then*

$$A \in \mathcal{D}_n \quad \Rightarrow \quad A/A_{11} \in \mathcal{D}_{n-k}.$$

In other words, if A is diagonally dominant, so is a Schur complement in A.

Proof. Let I_ϵ be a diagonal matrix with diagonal entries ϵ_i such that

$$|a_{ii} + \epsilon_i| > |a_{ii}|$$

for all $i = 1, 2, \ldots, n$. Let $A_\epsilon = A + I_\epsilon$. Then $A_\epsilon \in \mathcal{SD}_n$ since $A \in \mathcal{D}_n$.

Use A_ϵ in place of A in the proof of Theorem 3. A continuity argument yields the desired statement. □

Remark 1. As is known, the inverse of a strictly (row) diagonally dominant matrix is strictly diagonally dominant of its (column) *entries* (see, e.g., [2, p. 125]); the inverse need not be strictly diagonally dominant in the usual sense. Thus we see if $A \in \mathcal{SD}_n$, then, by Theorem 3 and the identity $(A/A_{11})^{-1} = (A^{-1})_{22}$, the Schur complements A/A_{11} and $A^{-1}/(A^{-1})_{11}$ are both strongly diagonally dominant of (column) entries. That is, if we denote $A/A_{11} = (\omega_{ij})$ and $A^{-1}/(A^{-1})_{11} = (\gamma_{ij})$, then for all i and j, $i \neq j$,

$$|\omega_{ij}| < |\omega_{jj}|, \quad |\gamma_{ij}| < |\gamma_{jj}|.$$

Remark 2. We summarize that the following classes are closed under taking the principal submatrices and the Schur complements: Positive semidefinite matrices; Hermitian matrices; M-matrices; inverse M-matrices; H-matrices; strictly diagonally dominant matrices; strictly double-diagonally dominant matrices; P-matrices; Note that the Schur complement of Z-matrices (the matrices all whose off-diagonal entries are nonpositive) need not be a Z-matrix.

Bibliography

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