Remarks on generalized numerical ranges of operators on indefinite inner product spaces

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Abstract. Numerical ranges associated to operators on an indefinite inner product space are investigated. Boundary generating curves, shapes, corners and computer generation of these sets are studied. Some final remarks present an interesting relation between these sets and numerical ranges of operators arising in quantum mechanics.

1 Introduction

The *numerical range* of a bounded linear operator A on a complex Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)$, is defined by

$$W(A) = \{(Ax, x) : x \in \mathcal{H},\ (x, x) = 1\}.$$

One of the most fundamental properties of the numerical range is its convexity, stated by the famous Toeplitz-Hausdorff Theorem [7, 9]. Other important property of $W(A)$ is that its closure contains the spectrum of the operator. $W(A)$ is a connected set and, in the finite dimensional case, is compact.

The theory of numerical ranges and its variations is rich and varied. A lot of recent research has been focused on the numerical ranges of operators on indefinite

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inner product spaces.

Let M_n be the algebra of $n \times n$ complex matrices and $S \in M_n$ be a Hermitian matrix. The S-*numerical range* of $A \in M_n$ is denoted and defined by

$$V_S(A) = \left\{ \frac{x^*Ax}{x^*Sx} : \ x \in \boldsymbol{C}^n, \ x^*Sx \neq 0 \right\}.$$

Consider the sets

$$V_S^{\pm}(A) = \{x^*Ax : \ x \in \boldsymbol{C}^n, \ x^*Sx = \pm 1\},$$

which have been studied by other researchers [13].

It is easy to verify that $V_{-S}^{+}(A) = V_S^{-}(A)$ and $V_S(A) = V_S^{+}(A) \cup -V_{-S}^{+}(A)$. If S is the identity matrix I_n, then $V_S(A) = V_S^{+}(A)$ reduces to the classical numerical range $W(A)$.

If S is a nonsingular indefinite Hermitian matrix, some authors use the sets $W_S(A) = V_S(SA)$ or $W_S^{+}(A) = V_S^{+}(SA)$ as the definition of a numerical range of a matrix A associated with the indefinite inner product $\langle x, y \rangle_S = y^*Sx$.

Motivated by theory and applications, there are several generalizations of the classical numerical range, such as the C-*numerical range* of A denoted by $W_C(A)$. For $A, C \in M_n$, $W_C(A)$ is defined by

$$W_C(A) = \{\mathrm{Tr}(CU^*AU) : \ U \in M_n, \ U^*U = I\}.$$

This concept motivates the definition of the S, C-*tracial range* of A for $A, C \in M_n$:

$$V_{S,C}(A) = \{\mathrm{Tr}(CU^*AU) : \ U \in M_n, \ U^*SU = S\}. \tag{1}$$

As a variation of (1), for $A, C \in M_n$ we define the S, C-*determinantal range* of A:

$$D_{S,C}(A) = \{\det(C + U^*AU) : \ U \in M_n, \ U^*SU = S\}.$$

When $S = I_n$, this concept reduces to the C-*determinantal range* $\triangle_C(A)$ [1].

When $C = \mathrm{diag}(\gamma_1, \ldots, \gamma_n)$, $W_C(A)$ and $\triangle_C(A)$ are usually denoted by $W_c(A)$ and $\triangle_c(A)$, respectively, where $c = (\gamma_1, \ldots, \gamma_n) \in \boldsymbol{C}^n$. The notations $V_{S,c}(A)$ and $\triangle_{S,c}(A)$ will be used in a similar way.

A matrix U in M_n is *pseudo-unitary of signature* $(r, n-r)$, with $0 \leq r \leq n$, if the corresponding linear transformation preserves the quadratic Hermitian form

$$q(x) = |x_1|^2 + \cdots + |x_r|^2 - |x_{r+1}|^2 - \cdots - |x_n|^2.$$

The group of pseudo-unitary matrices of signature $(r, n-r)$ will be denoted by $U_{r,n-r}$ and is connected. Since $V_{J,C}(A)$ $(D_{J,C}(A))$ is the range of the continuous mapping from $U_{r,n-r}$ to $\boldsymbol{C}$ defined by $U \mapsto \mathrm{Tr}(CU^*AU)$ $(U \mapsto \det(C + U^*AU))$, we conclude that $V_{J,C}(A)$ $(D_{J,C}(A))$ is a connected set, for any $A, C \in M_n$.

Let $0 \leq r \leq n$ and $J = P(I_r \oplus -I_{n-r})P^T$, where P is a permutation matrix. We easily verify that $U \in U_{r,n-r}$ if and only if $U^*JU = J$.

If we assume that the Hermitian matrix S is nonsingular, then it is not a restriction to consider the matrix J instead of S in the definition of the S-numerical range. In fact, recalling Sylvester's law of inertia [8], we can always choose a nonsingular matrix R, such that $R^*SR = I_r \oplus -I_{n-r}$, the inertia matrix of S and so $V_S(A) = V_J(A_R)$, for $A_R = R^*AR$ and J the inertia matrix of S. In an analogous way, the relations $V_{S,C}(A) = V_{J,C_{(R^*)^{-1}}}(A_R)$ and $D_{S,C}(A) = |\det R|^{-2}\, D_{J,C_R}(A_R)$, where $C_R = R^*CR$, hold.

2 Basic Properties

We list some basic properties of the S-numerical range that follow easily from the definition:

P1. $V_S(A) = V_S(U^*AU)$, for any $U \in M_n$ and any nonsingular Hermitian S such that $U^*SU = S$.

P2. $V_S(\alpha S + \beta A) = \alpha + \beta V_S(A)$, for any $\alpha, \beta \in \boldsymbol{C}$.

P3. $V_S(A^*) = \overline{V_S(A)}$.

P4. $V_S(A + B) \subset V_S(A) + V_S(B)$.

P5. $V_S(A) = \{\lambda\}$ if and only if $S \neq 0$ and $A = \lambda S$, for $\lambda \in \boldsymbol{C}$, that is, A is a *S-scalar matrix*.

P6. $V_S(A) \subseteq \boldsymbol{R}$ if and only if A is Hermitian.

The same properties are valid when V_S is replaced by $V_S^{\pm}$, except P2 and P5. For these sets, the following properties hold:

P2'. $V_S^{\pm}(\alpha S + \beta A) = \pm\alpha + \beta V_S^{\pm}(A)$, for any $\alpha, \beta \in \boldsymbol{C}$.

P5'. $V_S^{\pm}(A) = \{\lambda\}$ if and only if $A = \pm\lambda S$, for $\lambda \in \boldsymbol{C}$, and S has at least one positive (or negative) eigenvalue.

Denoting by $\sigma_S(A)$ the set of the eigenvalues of A that have *S-anisotropic* eigenvectors, that is, vectors x for which $x^*Sx \neq 0$, for S invertible, we obtain

P7. $\sigma_S(S^{-1}A) \subset V_S(A)$.

If $S = I_n$, then $\sigma_S(S^{-1}A) = \sigma(A)$, the spectrum of A, and the previous property reduces to the spectral inclusion of the classical numerical range.

If S is a nonsingular indefinite Hermitian matrix, and if A is not a S-scalar matrix, $V_S^+(A)$ is unbounded and may not be closed [13, 14]. The set $V_S^+(A)$ is convex; however $V_S(A)$ may not be convex. Nevertheless, $V_S(A)$ is *p-convex* [13]; that is, for any pair of distinct points $x, y \in V_S(A)$, either $V_S(A)$ contains the closed line segment joining x and y, or $V_S(A)$ contains the line defined by x and y, except the open line segment joining x and y.

The following properties concerning the S,C-tracial range and the S,C-determinantal range can be easily verified:

Q1. $V_{S,C}(A) = V_{S,C}(U^*AU)$ and $D_{S,C}(A) = D_{S,C}(U^*AU)$, for any $U \in M_n$ and any nonsingular Hermitian S such that $U^*SU = S$.

Q2. $V_{S,C}(\alpha S + \beta A) = \alpha \operatorname{Tr}(SC) + \beta V_{S,C}(A)$, for any $\alpha, \beta \in \boldsymbol{C}$.

Q3. $V_{S,C^*}(A^*) = \overline{V_{S,C}(A)}$ and $D_{S,C^*}(A^*) = \overline{D_{S,C}(A)}$, for S nonsingular Hermitian.

Q4. $V_{S,C}(A) = V_{S^{-1},A}(C)$, for S nonsingular Hermitian.

Q5. $D_{S,C}(A) = D_{S,A}(C)$, that is, the roles of A and C are symmetric.

In the theory of classical numerical ranges and its generalizations, the reduction of problems to its bidimensional case is a very useful technique. For instance, convexity results can be proved using such a reduction. In this theory, the elliptical range theorem is a particularly important result. It asserts that the classical numerical range of a 2×2 matrix A is an elliptical disc, possibly degenerate, with the eigenvalues λ_1 and λ_2 of A as foci and minor axis of length

$$\sqrt{\operatorname{Tr}(A^*A) - |\lambda_1|^2 - |\lambda_2|^2}.$$

Now, we give a detailed geometric description of the J-numerical range of $A \in M_2$.

Theorem 1 (Hyperbolical Range Theorem). [2] *Let $A = (a_{ij}) \in M_2$ and $J = \operatorname{diag}(1, -1)$. Let λ_1, λ_2 be the eigenvalues of JA,*

$$M = |\lambda_1|^2 + |\lambda_2|^2 - \operatorname{Tr}(A^*JAJ) \qquad \text{and} \qquad N = \operatorname{Tr}(A^*JAJ) - 2\operatorname{Re}(\overline{\lambda_1}\lambda_2).$$

Denote by l the line passing through $\lambda = \frac{1}{2}Tr(JA)$ and perpendicular to the line defined by λ_1 and λ_2.

a) *If $M > 0$ and $N > 0$, then $V_J(A)$ is bounded by the hyperbola with λ_1 and λ_2 as foci, transverse axis of length $\sqrt{N}$ and non-transverse axis of length $\sqrt{M}$.*

b) *If $M > 0$ and $N = 0$, then $V_J(A)$ is*

- **i)** *the line l, if $|a_{12}| = |a_{21}|$;*
- **ii)** *the whole complex plane except the line l, if $|a_{12}| \neq |a_{21}|$.*

c) *If $M > 0$ and $N < 0$, then $V_J(A)$ is the whole complex plane.*

d) *If $M = 0$ and $N > 0$, then $V_J(A)$ is the line defined by λ_1 and λ_2, except the open line segment joining λ_1 and λ_2.*

e) *If $M = N = 0$, then $V_J(A)$ is*

- **i)** *the singleton $\{\lambda\}$, if $\operatorname{Tr} A = 0$;*
- **ii)** *the line defined by a_{11} and $-a_{22}$, except the point λ, if $\operatorname{Tr} A \neq 0$.*

The next theorem characterizes $V_{J,c}(A)$ and $D_{J,c}(A)$ in the 2×2 case, for $J = \operatorname{diag}(1, -1)$. It can be easily deduced from the hyperbolical range theorem (for a proof see [2]).

Theorem 2. *Let $c = (\gamma_1, \gamma_2) \in \boldsymbol{C}^2$, $A \in M_2$ and $J = \operatorname{diag}(1, -1)$. Let λ_1 and λ_2 be the eigenvalues of JA. Then*

i) *$V_{J,c}(A)$ is bounded by a branch of a (possibly degenerate) hyperbola, with foci $\gamma_1\lambda_1 - \gamma_2\lambda_2$ and $\gamma_1\lambda_2 - \gamma_2\lambda_1$;*

ii) *$D_{J,c}(A)$ is bounded by a branch of a (possibly degenerate) hyperbola, with foci $(\gamma_1 + \lambda_1)(\gamma_2 - \lambda_2)$ and $(\gamma_1 + \lambda_2)(\gamma_2 - \lambda_1)$.*

Both hyperbolas have non-transverse axis of length

$$|\gamma_1 + \gamma_2| \sqrt{|\lambda_1|^2 + |\lambda_2|^2 - \operatorname{Tr}(A^* JAJ)}.$$

For the degenerate cases, we may have a singleton, a line, a subset of a line, an open half plane or the whole complex plane.

3 Geometrical Properties

It is well known that the classical numerical range of A is a singleton if and only if A is a scalar matrix. Moreover $W(A) \subseteq \boldsymbol{R}$ if and only if A is Hermitian. Analogous results are valid for the J, c-tracial range, for $J = P(I_r \oplus -I_{n-r})P^t$, $0 \leq r \leq n$. The ith diagonal entry of J is denoted by j_i.

Theorem 3. *Let $c = (\gamma_1, \ldots, \gamma_n) \in \boldsymbol{C}^n$, where the $\gamma_i j_i$ are pairwise distinct, for $i = 1, \ldots, n$. For $A \in M_n$, $V_{J,c}(A)$ is a singleton if and only if A is J-scalar.*

Theorem 4. *Let $A \in M_n$ and $c = (\gamma_1, \ldots, \gamma_n) \in \boldsymbol{R}^n$, with the $\gamma_i j_i$ pairwise distinct, for $i = 1, \ldots, n$. Then $V_{J,c}(A) \subseteq \boldsymbol{R}$ if and only if A is Hermitian.*

A matrix $A \in M_n$ is said to be *essentially J-Hermitian* if $\mu A + \upsilon J$ is Hermitian for some $0 \neq \mu \in \boldsymbol{C}$ and $\upsilon \in \boldsymbol{C}$.

Theorem 5. *Let $c = (\gamma_1, \ldots, \gamma_n) \in \boldsymbol{R}^n$, with the $\gamma_i j_i$ pairwise distinct, for $i = 1, \ldots, n$, and $A \in M_n$. Then $V_{J,c}(A)$ is a subset of a straight line if and only if A is essentially J-Hermitian.*

Since $\triangle_C(A)$ may be seen as the range of a function from U_n to $\boldsymbol{C}$, $\triangle_C(A)$ may be considered a variation on the concept of $W_C(A)$. In fact, these two sets have many common properties. As will be seen in the next results, a certain parallelism still exists between the sets $V_{J,c}(A)$ and $D_{J,c}(A)$.

Theorem 6. *Let $c = (\gamma_1, \ldots, \gamma_n) \in \boldsymbol{C}^n$, where the $\gamma_i j_i$ are pairwise distinct, for $i = 1, \ldots, n$, and $A = \operatorname{diag}(\alpha_1, \ldots, \alpha_n) \in M_n$. Then $D_{J,c}(A)$ is a singleton if and only if A is J-scalar.*

Theorem 7. *Let* $c = (\gamma_1, \ldots, \gamma_n) \in \boldsymbol{R}^n$*, where the* $\gamma_i j_i$ *are pairwise distinct, for* $i = 1, \ldots, n$*, and* $A = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$*. Then* $D_{J,c}(A) \subseteq \boldsymbol{R}$ *if and only if* $A \in M_n(\boldsymbol{R})$*.*

Let $z \in V_S(A)$ be a boundary point of $V_S(A)$. A line containing z and defining two half planes, such that one of them does not contain $V_S^+(A)$ or $-V_{-S}^+(A)$ will be called a *support line* of $V_S(A)$.

Murnaghan [18] and Kippenhahn [12], independently, showed that the boundary of the classical numerical range of a matrix $A \in M_n$ is the set of real points of the algebraic curve (of class n) with equation in line coordinates

$$\det(uH + vG + wI_n) = 0,$$

where H and G are Hermitian matrices satisfying $A = H + iG$. In [2], we generalize this result for the S-numerical range, where S is a nonsingular Hermitian matrix.

Theorem 8. *Let* $ux + vy + w = 0$ *be the equation of a support line of* $V_S(A)$ *and let* $A = H + iG$*, where* H *and* G *are Hermitian matrices. Then* $\det(uH + vG + wS) = 0$.

Some observations are in order:

1. Since $\det(uH + vG + wS)$ is a homogeneous polynomial of degree n, the equation

$$\det(uH + vG + wS) = 0, \tag{2}$$

with u, v, w viewed as homogeneous line coordinates, defines an algebraic curve of class n, whose real part, denoted by $C_S(A)$, forms the boundary of $V_S(A)$. This curve has class n and has real coefficients, thus it has n real foci [20, 21], corresponding to the eigenvalues of the matrix $S^{-1}A$.

2. The dual curve of a conic is again a conic. Hence, if A is a 2×2 matrix, the equation (2) defines a conic and so its real part is a hyperbola, a parabola or an ellipse, possibly degenerate. If A is a S-scalar matrix, then by property P5, $V_S(A)$ is a singleton. Otherwise, if S is nonsingular indefinite, then $V_S(A)$ is unbounded and p-convex, hence $C_S(A)$ must be an hyperbola, whose foci are the eigenvalues of $S^{-1}A$. Obviously, $V_S(A)$ consists of the hyperbola and its interior. If S is definite, then $V_S(A)$ is bounded and convex, therefore the convex hull of $C_S(A)$ is an elliptical disc, with the eigenvalues of $S^{-1}A$ as foci.

3. If the inertia matrix of S is I_n, then Theorem 8 reduces to the Murnaghan-Kippenhahn Theorem.

Using a theorem of Tarski [11], it was shown in [6] that the boundary of $\triangle_C(A)$ is a finite union of algebraic arcs. The same result holds for $W_C(A)$ (see [19]). Using Tarski theorem, we can prove that the boundaries of $V_{S,C}(A)$ and $D_{S,C}(A)$ are also finite unions of algebraic arcs.

Theorem 9. [2] *Let* $A, C \in M_n$*. The boundary of* $V_{S,C}(A)$ *and the boundary of* $D_{S,C}(A)$ *are a finite union of algebraic arcs, and so they are curves of class* $\boldsymbol{C}^\infty$*, except for a finite number of points.*

A boundary point μ of a subset K in $\boldsymbol{C}$ is a *corner* of K if there exists a sufficiently small $\epsilon > 0$ such that the intersection of K and the circular disc $\mathcal{D} = \{v \in \boldsymbol{C} : |v - \mu| < \epsilon\}$ is contained in a sector of $\mathcal{D}$ of degree less than π.

Some special boundary points of the numerical ranges have interesting properties, such as: there is an interplay between the geometrical properties of the numerical range and the algebraic properties of the operator. For instance, every non-differentiable boundary point of $W(A)$ is an eigenvalue of A [5, 12]. Also, every corner of $V_S(A)$ is an eigenvalue of $S^{-1}A$ [15]. For $V_{J,c}(A)$ and $D_{J,c}(A)$, the following holds:

Theorem 10. [2] *Let A and C be matrices in M_n, such that*

$$CJ = \gamma_1 I_{n_1} \oplus \cdots \oplus \gamma_p I_{n_p}, \qquad n_1 + \cdots + n_p = n,$$

where the γ_i are pairwise distinct, for $i = 1, \dots, p$, and let $U \in U_{r,n-r}$.

- **i)** *If $z = \operatorname{Tr}(CU^*AU)$ is a corner of $V_{J,C}(A)$, then $U^*AU = A_1 \oplus \cdots \oplus A_p$, where $A_i \in M_{n_i}$, $i = 1, \dots, p$, and $z = \sum_{i=1}^{p} \gamma_i \operatorname{Tr}(J_i A_i)$, where $J_i \in M_{n_i}$, $i = 1, \dots, p$, are such that $J = J_1 \oplus \cdots \oplus J_p$.*
- **ii)** *If $0 \neq w = \det(C + U^*AU)$ is a corner of $D_{J,C}(A)$, then $U^*AU = A_1 \oplus \cdots \oplus A_p$, where $A_i \in M_{n_i}$, $i = 1, \dots, p$, and $w = \prod_{i=1}^{p} \det(\gamma_i J_i + A_i)$, where $J_i \in M_{n_i}$, $i = 1, \dots, p$, are such that $J = J_1 \oplus \cdots \oplus J_p$.*

The following result is due to Li and Rodman [15] and is a straightforward consequence of Theorem 10 i).

Corollary 11. *Let A be a matrix in M_n. If $z \in V_J(A)$ is a corner of $V_J(A)$, then z is an eigenvalue of JA and there exists an eigenvector x associated to z, such that $A\,x = z\,Jx$, $A^*\,x = \bar{z}\,J\,x$ and $x^*Jx = \pm 1$.*

The hyperbolical range theorem for the J, c-tracial range of 2×2 matrices leads to the characterization of those matrices $A \in M_n$ for which $V_{J,c}(A)$ is a singleton or a subset of a straight line. This result is also useful in the study of some special boundary points of the J, c-tracial range, as those in the following theorem.

Theorem 12. *Let $c = (\gamma_1, \dots, \gamma_n) \in \boldsymbol{C}^n$ and $A = (a_{kl})$ be an upper triangular matrix in M_n, with diagonal elements $\alpha_1, \dots, \alpha_n$. If $\sum_{i=1}^{n} \gamma_i \alpha_i$ is a boundary point of $V_{J,c}(A)$ and $\gamma_k j_k \neq \gamma_l j_l$, then $a_{kl} = 0$, for $1 \leq k < l \leq n$.*

The following corollaries are easy consequence of this theorem:

Corollary 13. *Let A be an upper triangular matrix in M_n, and let*

$$CJ = \gamma_1 I_{n_1} \oplus \cdots \oplus \gamma_p I_{n_p}, \qquad n_1 + \cdots + n_p = n,$$

where the $\gamma_i j_i$ are pairwise distinct, for $i = 1, \dots, p$. If $\operatorname{Tr}(CA)$ is a boundary point of $V_{J,C}(A)$, then $A = A_1 \oplus \cdots \oplus A_p$, where each block A_i is an upper triangular matrix in M_{n_i}, $i = 1, \dots, p$.

Corollary 14. *Let* $c = (\gamma_1, \ldots, \gamma_n) \in \boldsymbol{C}^n$*, with the* $\gamma_i j_i$ *pairwise distinct, for* $i = 1, \ldots, p$*, and let* A *be an upper triangular matrix in* M_n *with diagonal elements* $\alpha_1, \ldots, \alpha_n$*. If* $\sum_{i=1}^{n} \gamma_i \alpha_i$ *is a boundary point of* $V_{J,c}(A)$*, then* A *is a diagonal matrix.*

4 Computer Generation of $V_S(A)$

Many algorithms and computer programs for generating the classical numerical range and its generalizations have been presented [11, 17].

In [14], a Matlab program for generating $W_S^+(A) = V_S^+(SA)$, for S a Hermitian matrix, has been developed. We present an algorithm providing the boundary generating curve of the J-numerical range. Matlab programs maybe developed to plot this curve and to draw an approximation for $V_J(A)$.

Step 1: Consider the directions $e^{i\theta_s}$, where

$$\theta_s = \frac{\pi(s-1)}{2m}, \qquad s = 1, \ldots, 4m+1,$$

for some positive integer $m \geq 20$. For $A \in M_n$ and for each choice of s, compute the eigenvalues of B_s. If these eigenvalues are all real and the norm of the corresponding eigenvectors is not 0, put the value s in a vector denoted by *direc* $= (t_1, \ldots, t_k)$. We observe that the number of eigenvectors with positive J-norm is fixed by the inertia of J, independently of θ_s.

Step 2: If the vector *direc* is empty, the following cases may occur:

(i) If A is Hermitian, then $V_J(A) = \boldsymbol{R}$;

(ii) If A is essentially J-Hermitian, then $V_J(A)$ is a line;

(iii) Otherwise, $V_J(A)$ is the whole complex plane, possibly without a line.

Step 3: If the situations referred to in the previous step do not occur, then for each entry t_j of the vector *direc*, compute n linearly independent eigenvectors associated with the eigenvalues of B_{t_j}. (The linear independence of the eigenvectors ensures that all the generating boundary curves are considered.) Denote these eigenvectors by $u_i(t_j)$, for $i = 1, \ldots, n$, and evaluate

$$\frac{u_i^*(t_j) A\, u_i(t_j)}{u_i^*(t_j) J\, u_i(t_j)},$$

for $u_i^*(t_j) J\, u_i(t_j) > 0$ and $u_i^*(t_j) J\, u_i(t_j) < 0$.

To generate an approximation of $V_J(A)$ we proceed as follows.

Step 4: For any distinct points $x, y \in V_J^+(A)$, compute $\alpha x + (1-\alpha) y$, for $0 \leq \alpha \leq 1$. Repeat this step for $-V_J^-(A)$.

Step 5: For any distinct points $x \in V_J^+(A)$ and $y \in -V_J^-(A)$, compute $\alpha x + (1-\alpha) y$, for $\alpha \leq 0$ or $\alpha \geq 1$.

The set obtained in Step 4 and 5 is the *pseudo-convex hull* of the boundary generating curves of $V_J(A)$.

5 Final Remarks

Let $\{e_1, e_2\}$ be an orthonormal basis for $\boldsymbol{C}^2$ and $m \in \boldsymbol{N}_0$. The mth completely symmetric space over $\boldsymbol{C}^2$, denoted by $\boldsymbol{C}^2_{(m)}$, is spanned by the vectors $e_1^k * e_2^{m-k}$, for $k = 0, \ldots, m$, where e_i^k denotes the symmetric tensor product $e_i * \cdots * e_i$ with k factors.

In quantum mechanics, creation and annihilation operators are the building blocks of linear operators acting on Hilbert spaces of many-body systems [4]. In the bosonic case, the *creation operator* $f_i : \boldsymbol{C}^2_{(m)} \to \boldsymbol{C}^2_{(m+1)}$, $i = 1, 2$, is the linear operator defined by $f_i(x^*) = e_i * x^*$, for $x^* = x_1 * \cdots * x_m$ a decomposable tensor in $\boldsymbol{C}^2_{(m)}$. The *annihilation operator* $g_i : \boldsymbol{C}^2_{(m+1)} \to \boldsymbol{C}^2_{(m)}$ is just the adjoint operator of the creation operator f_i. These operators satisfy the *canonical commutation relations* $[f_i, f_j] = [g_i, g_j] = 0$, $[g_i, f_j] = \delta_{ij}$, $i, j = 1, 2$, where $[f, g] = fg - gf$. The creation and annihilation operators can also be defined on the symmetric algebra over $\boldsymbol{C}^2$,

$$\Gamma^* = \bigoplus_{m=0}^{+\infty} \boldsymbol{C}^2_{(m)}.$$

We consider Γ^* endowed with the norm induced by the standard inner product defined for all decomposable tensors $x^* = x_1 * \cdots * x_m$ and $y^* = y_1 * \cdots * y_m$ in $\boldsymbol{C}^2_{(m)}$ by $(x^*, y^*) = \mathrm{per}[(x_i, y_j)]$, where $\mathrm{per}X$ denotes the permanent of the matrix X.

The *pairing operator* $B : \Gamma^* \to \Gamma^*$ is the linear operator defined by

$$B = c\, f_1 g_1 + d\, f_2 g_2 + k\, f_1 f_2 + l\, g_1 g_2,$$

with c, d, k, l complex numbers. The classical numerical range of pairing operators has an interesting relation with the J-numerical range.

Denote by $\Gamma^{(q)}$, for $q \geq 0$, the subspace of the symmetric algebra over $\boldsymbol{C}^2$ spanned by the vectors $e_1^n * e_2^{n+q}$, $n \in \boldsymbol{N}_0$, and, for $q < 0$, the subspace spanned by the vectors $e_1^{n-q} * e_2^n$, $n \in \boldsymbol{N}_0$. We easily check that the subspace $\Gamma^{(q)}$ is invariant under the pairing operator B, that is, $B(\Gamma^{(q)}) \subseteq \Gamma^{(q)}$, for any $q \in \boldsymbol{Z}$. It is also clear that

$$\Gamma^* = \bigoplus_{q=-\infty}^{+\infty} \Gamma^{(q)}.$$

For simplicity, let the pairing operator B be restricted to $\Gamma^{(0)}$ (if $q \neq 0$, the study follows along similar lines).

Let

$$W = \left\{ \frac{(c+d)|z|^2 + k\bar{z} + lz}{1 - |z|^2} \; : \; z \in \boldsymbol{C},\ |z| < 1 \right\}.$$

The hyperbolical range theorem provides the characterization of the set W. In fact, it can be easily verified that $W = V_J^+(A)$, where $J = \mathrm{diag}(1, -1)$ and

$$A = \begin{bmatrix} 0 & l \\ k & c+d \end{bmatrix}.$$

It can be proved that $W(B|_{\Gamma^{(0)}}) = W$ [3], and so the following result holds.

Theorem 15. *Let the pairing operator $B = c\, f_1 g_1 + d\, f_2 g_2 + k\, f_1 f_2 + l\, g_1 g_2$, with $c, d, k, l \in \boldsymbol{C}$ be restricted to $\Gamma^{(0)}$. Let*

$$M = \frac{|\Delta| + P}{2} \qquad \text{and} \qquad N = \frac{|\Delta| - P}{2},$$

where

$$\Delta = (c+d)^2 - 4kl \qquad \text{and} \qquad P = |c+d|^2 - 2|k|^2 - 2|l|^2.$$

Denote by r the line passing through $-\frac{1}{2}(c+d)$ and perpendicular to the line defined by

$$\alpha_1 = -\frac{1}{2}(c+d) + \frac{1}{2}\sqrt{\Delta} \qquad \text{and} \qquad \alpha_2 = -\frac{1}{2}(c+d) - \frac{1}{2}\sqrt{\Delta}.$$

Denote by s the line defined by 0 and $c+d$.

- **a)** *If $M > 0$ and $N > 0$, then $W(B|_{\Gamma^{(0)}})$ is bounded by a branch of hyperbola with α_1 and α_2 as foci, transverse and non-transverse axis of length $\sqrt{N}$ and $\sqrt{M}$, respectively.*
- **b)** *If $M > 0$ and $N = 0$, then $W(B|_{\Gamma^{(0)}})$ is*
  - **i)** *the line r, if $|k| = |l|$;*
  - **ii)** *a open-half plane defined by the line r, if $|k| \neq |l|$.*
- **c)** *If $M > 0$ and $N < 0$, then $W(B|_{\Gamma^{(0)}})$ is the whole complex plane.*
- **d)** *If $M = 0$ and $N > 0$, then $W(B|_{\Gamma^{(0)}})$ is a closed-half line contained in s with endpoint α_1 or α_2.*
- **e)** *If $M = N = 0$, then $W(B|_{\Gamma^{(0)}})$ is*
  - **i)** *the singleton $\{0\}$, if $c + d = 0$;*
  - **ii)** *a open-half line contained in s with endpoint $-\frac{1}{2}(c+d)$, if $c + d \neq 0$.*

The matrix representation of the Hermitian pairing operator B, restricted to the subspace $\Gamma^{(q)}$ of the symmetric algebra over $\boldsymbol{C}^2$, in the standard basis, is an infinite tridiagonal matrix, thus Theorem 15 also gives the characterization of the numerical range of a infinite tridiagonal matrix.

There are other interesting relations in the same vein between the classical numerical range and the J-numerical range that deserve investigation.

Bibliography

[1] N. Bebiano, *Some analogies between the c-numerical range and a certain variation on this concept*, Linear Algebra Appl., 81 (1986), 47-54.

[2] N. Bebiano, R. Lemos, J. da Providência and G. Soares, *On generalized numerical ranges of operators on an indefinite inner product space*, submitted.

[3] N. Bebiano, R. Lemos and J. da Providência, *Numerical ranges of operators arising in quantum mechanics*, preprint.

[4] J. P. Blaizot and G. Ripka, *Quantum Theory of Finite Systems*, The MIT Press, Cambridge, 1986.

[5] W. F. Donoghue, *On the numerical range of a bounded operator*, Michigan Math. J., 4 (1957), 261-263.

[6] A. L. Duarte, *O Teorema de Tarski e suas Aplicações em Teoria de Matrizes*, Universidade de Coimbra, 1993.

[7] K. Gustafson and D. Rao, *Numerical Range - The field of values of linear operators and matrices*, Springer, New York, 1997.

[8] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, New York, 1985.

[9] ——, *Topics in Matrix Analysis*, Cambridge University Press, New York, 1991.

[10] N. Jacobson, *Basic Algebra*, Freeman, 1979.

[11] C. R. Johnson, *Numerical determinantion of the field of values of a general complex matrix*, SIAM J. Numer. Anal, 15 (1978), 595-602.

[12] R. Kippenhahn, *Uber der wertevorrat einer matrix*, Math. Nach., 6 (1951), 193-228.

[13] C.-K. Li, N. K. Tsing and F. Uhlig, *Numerical ranges of an operator in an indefinite inner product space*, Electr. J. Linear Algebra, 1 (1996), 1-17.

[14] C.-K. Li and L. Rodman, *Shapes and computer generation of numerical ranges of Krein space operators*, Electr. J. Linear Algebra, 3 (1998), 31-47.

[15] ——, *Remarks of numerical ranges of operators in spaces with an indefinite inner product*, Proc. Amer. Math. Soc., 126 (1998), 973-982.

[16] M. Marcus, *Finite Dimensional Multilinear Algebra I*, Marcel Dekker, New York, 1973.

[17] M. Marcus and C. Pesce, *Computer generated numerical ranges and some resulting theorems*, Linear and Multilinear Algebra Appl., 20 (1987), 121-157.

[18] F. D. Murnaghan, *On the field of values of a square matrix* Proc. Nat. Acad. Sci., 18 (1932), 246-248.

[19] H. Nakazato, Y. Nishikawa and M. Takaguchi, *On the boundary of the C-numerical range of a matrix*, Linear Algebra Appl., 39 (1995), 231-240.

[20] H. Shapiro, *A conjecture of Kippenhahn about the characteristic polynomial of a pencil generated by two Hermitian matrices. I*, Linear Algebra Appl., 43 (1982), 201-221.

[21] ——, *A conjecture of Kippenhahn about the characteristic polynomial of a pencil generated by two Hermitian matrices. II*, Linear Algebra Appl., 45 (1982), 97-108.