A Novel Technique for Simplifying Multi-scale, Linear Fluid Dynamics Problems

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Abstract: Of fundamental importance in any numerical computation of fluid dynamics problems is the optimal, low dimensional representation of an essentially infinite dimensional dynamical system phenomena. In this paper, we introduce a novel technique for getting simple models of unsteady fluid phenomena. The main idea behind this method is deleting the weakly controllable and weakly observable modes of the fluid after the controllability and the observability gramians of the fluid are aligned through a similarity transformation. Computations done on Couette flow using spectral method, Fourier in span wise direction and Chebeshev collocation in wall normal direction will be presented.

1 Introduction

The governing equations of fluid mechanics, Navier-Stokes equations, are a set of coupled partial differential equations. Central to any numerical simulation is, the problem of representing these partial differential equations by finite set of ordinary differential equations. This process is achieved through some projection technique. Numerical simulations of these extremely large number of finite dimensional equations - of the order of few thousands - are very expensive, both from computational time and memory. Hence, it is of considerable interest to project the dynamics of these large number of ordinary differential equations onto a proper low dimensional subspace of few ordinary differential equations.

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This low dimensional representation of a physical phenomena is also important from another point of view. It is of great interest to see what the important modes in any physical phenomena are. This might lead to the better understanding of the underlying dynamics and physics involved.

The traditional methods used for reducing the dimensions of fluid mechanics problems are Karhunen-Loeve decomposition or Principal orthogonal decomposition (POD) [5, 6] and Singular perturbation technique. POD was introduced by Lumley [7, 4] into turbulence. The essential idea in POD is the projection of the dynamics of the system onto few basis functions which have the optimal energy in the L_2 sense. Singular perturbation is an time scale separation technique, which projects the dynamics onto a slow manifold by truncating the fast manifold dynamics.

Even though for some applications the most energetic modes are the important modes, it need not be the case always [2]. The most important thing in any problem is, what is driving the system (input), and what is it that one is interested in (output). We will argue that capturing this input-output behavour is very important. In this paper we introduce a new complexity reduction technique into fluids, which takes into account the underlying input-output properties of fluids. This is based on [8]. This method has considerable advantages like rigorous error bounds [3] and transparent physics. The physics becomes more clear through the use of the new concepts like controllability and observability. The error is quantified in terms of the H_∞ norm of the difference of the unreduced and reduced transfer functions.

This paper is structured as follows. In the next section we briefly discuss the notation used in this paper. The following sections are devoted to explaining the basic idea of the method and the concepts involved. After this, we briefly explain the numerical details and present the results of computations done on linearized Couette flow. This is followed by conclusions and future work.

2 Notation

In this section we will discuss the notation that we will be using in this paper. The discretized linearized Navier-Stokes equations can be written in the following form.

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t) \qquad (1)\\ y(t) &= Cx(t) \\ x(0) &= x_0 \end{aligned}$$

where $x(.) : R \mapsto X$ is the state of the system, $x_0 \in X$ is the initial condition, $w(.) : R \mapsto W$ is the disturbance driving the fluid and $y(.) : R \mapsto Y$ is the output, i.e., the quantity we are interested in. $A \in L(X;X)$, $B \in L(W;X)$, and $C \in L(X;Y)$ are bounded linear operators. The spaces X, W and Y are the state space, disturbance space and output space respectively, and they are assumed to be linear finite dimensional vector spaces. In this paper $X = C^n$, $W = C^m$, $Y = C^k$. A, B and C have appropriate dimensions. C^n denotes the complex n dimensional vector space. We will assume that the operator A is Hurwitz as we need to solve some Lyapunov equations[1]. Taking the Laplace transforms with zero initial conditions of

[1]Lyapunov equations have unique solution only when A is Hurwitz

(1) we get the frequency domain characterization of the system.

$$\hat{y}(s) = C(sI - A)^{-1}B\hat{w}(s) \equiv G(s)\hat{w}(s) \tag{2}$$

$G(s)$ is called the transfer function of the fluid. The norm of the matrix will be denoted by $\| \ \|$ with the appropriate subscript. For example, $\| \ \|_2$ denotes the standard Eucledian 2 norm. * denotes conjugate transpose. The H_∞ norms is defined as

$$\|G\|_{H_\infty} \equiv \sup_{\omega \in R} \bar{\sigma}[G(j\omega)] \tag{3}$$

where $j = \sqrt{-1}$ and $\bar{\sigma}[M]$ denotes the maximum singular value of the operator M.

3 Complexity Reduction

Given a large system of equations (1) with the transfer function $G(s)$. We would like to approximate the input-output characteristics of this large dynamical system with another transfer function $G_r(s)$ which has less complexity $r << n$. The complexity is measured here in terms of the state space dimensions r and n of original (1) and truncated fluid (4) equations. The state space representation of the truncated fluid is given by

$$\begin{aligned} \dot{x_r}(t) &= A_r x_r(t) + B_r w(t) \\ y(t) &= C_r x_r(t) \\ x_r(0) &= x_{r_0} \end{aligned} \tag{4}$$

$G_r(s) = C_r(sI - A)^{-1}B_r$ is the transfer function of the truncated fluid. $X_r = C^r$, $W = C^m$, $Y = C^k$, $A_r \in C^{r \times r}$, $B_r \in C^{r \times m}$ and $C_r \in C^{k \times r}$. The error $\|G - G_r\|$ made in the approximation will be measured in terms of the H_∞ norm.

The basic idea behind this method is deleting the weakly controllable and weakly observable modes of the flow, after the controllability and the observability gramians of the flow are aligned through a similarity transformation. In the next subsections, we discuss the details of controllability and observability operators and their respective gramians.

3.1 Controllability Operator and Gramian

Given input-output representation (1) of the unsteady flow phenomena. There are only certain places the fluid can flow or reach in the state space with a given input structure. These states are called the reachable states or controllable states. The states that cannot be reached with a given input structure are called unreachable or uncontrollable states. Understanding these reachable subspaces is important because, then, one can truncate the unreachable subspaces as the system can never go there. Below we give a rigorous characterization of the reachable and un reachable states of the fluid (1). Taking $C = 0$ in (1) and prescribing the initial conditions at $t = -\infty$, we get

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bw(t) \\ x(-\infty) &= 0 \end{aligned} \tag{5}$$

With a given input structure $w(t)$, we want to see what all are the possible states $x(0)$ that can be reached. It follows from (5) that, x(0) is given by

$$x(0) = \int_{-\infty}^{0} e^{-A\tau} Bw(\tau)d\tau \equiv O_c w(t) \tag{6}$$

Where, O_c is called the controllability map

$$O_c : L_2(-\infty, 0] \to C^n \tag{7}$$
$$w(t) \mapsto x_0 \tag{8}$$

It is a map from past input to state of the flow at $t = 0$. Next we would like to address the question: what all are the states $x(0)$, that are accessible with given input, such that, $w(t) \in L_2(-\infty, 0]$ and $\|w(t)\|_{L_2} \le 1$. That is we want to characterize the set

$$R_0 = (x(0) = O_c w(t) \ : \ w(t) \in L_2(-\infty, 0], \ \|w(t)\|_{L_2} \le 1) \tag{9}$$

It can be shown that [8, 9, 1]

$$R_0 = \left(X_c^{\frac{1}{2}} z : z \in C^n \ and \ \|z\|_2 \le 1 \right) \tag{10}$$

Where, X_c is called the controllability gramian. It is defined by

$$X_c = \int_{-\infty}^{0} e^{-A\tau} BB^* e^{-A^*\tau} d\tau \tag{11}$$

It is not hard to see that the relationship between controllability gramian X_c and the controllability operator is given by

$$X_c = O_c O_c^* \tag{12}$$

Where, O_c^* is the adjoint controllability operator defined by

$$O_c^* : C^n \to L_2(-\infty, 0] \tag{13}$$
$$r_0 \mapsto B^* e^{-A^* t} r_0 \tag{14}$$

The boundary of R_0 is given by

$$E_c = \left(X_c^{\frac{1}{2}} z : z \in C^n \ and \ \|z\|_2 = 1 \right) \tag{15}$$

We call this the controllability ellipsoid as $\|X_c^{\frac{1}{2}} z\|_2 = z^* X_c z$ and X_c is a positive definite matrix. The above set is made up of states ($x(0)$), that can be reached with $\|w(t)\|_{L_2} = 1$.

Let $\lambda_1 \ge \lambda_2 .. \ge \lambda_n \ge 0$ be the eigenvalues and $\chi_1 \ge \chi_2 .. \ge ....\chi_n$ be the orthonormal eigen vectors of $X_c^{\frac{1}{2}}$. The orthonormal eigen vectors of the controllability ellipsoid form the principal axis of the ellipsoid and they form an orthonormal basis of the flow state space. The eigen values essentially tell that, the maximum distance we can move in a certain direction χ_r is λ_r with an $\|w(t)\|_{L_2} \le 1$ input. Hence, $\lambda_r > \lambda_s$ means that χ_r is more easily reachable than χ_s or χ_r is more controllable than χ_s. In conclusion, controllability gramian carries the information about the set of the reachable or controllable states.

3.2 Observability Operator and Gramian

In many cases we are interested only in certain characteristics or output of the flow. Once the output is chosen, it can be influenced only by certain states of the flow. The rest of the states which do not have much influence on the output can be deleted with very small error on output characteristics. In this section we put the above physical picture in abstract terms. Taking $w(t) = 0$ in (1), we get

$$\begin{aligned} \dot{x}(t) &= Ax(t) \\ y(t) &= Cx(t) \\ x(0) &= x_0 \end{aligned} \tag{16}$$

Let's say, we have no knowledge of the initial condition of the fluid. We would like to ask, if it is possible, to observe the output for a finite time interval $[0, T]$, and then estimate the initial condition and hence the entire future state trajectory. The solution of eqn(16) is

$$y(t) = Ce^{At}x_0 \equiv O_o x_0 \tag{17}$$

Now, the initial condition can only be defined without ambiguity if the equation $y = O_o x_0$ has unique solution. This is possible if and only if $ker\ O_o = 0$. For a system with $ker\ O_o \neq 0$, there are ceratin states which are not observable from the output y. I.e., there are ceratin states which have no infuence on what the output is. Hence we call $ker\ O_o$ the unobservable subspace. The space orthogonal to this is called the observable subspace.

As in the previous case, we define the controllability operator and gramian, and show their relations to observable and unobservable modes. The observability operator is defined as map from intial conditions to output

$$\begin{aligned} O_o \ : \ C^n &\rightarrow L_2[0, \infty] \\ x_0 &\mapsto O_c x_0 \end{aligned} \tag{18}$$

The total energy in the output is given by

$$\|y(t)\|^2_{L_2} \quad = \quad < O_o x_o, O_o x_o > \quad = \quad < x_o, O_o^* O_o x_o > \tag{19}$$

$O_o^* \ : \ L_2[0, \infty \rightarrow]C^n$ is the adjoint observability operator and is given by

$$O_o^* v(t) = \int_0^\infty e^{A^*\sigma} C^* v(\sigma) d\sigma \tag{20}$$

Energy equation (19) can be written as

$$\|y(t)\|^2_{L_2} = x_0^* Y_o x_0 \tag{21}$$

$$Y_o = O_o^* O_o = \int_0^\infty e^{A^*\sigma} C^* C e^{A\sigma} d\sigma \tag{22}$$

Where, Y_o is called the observability gramian. Eqn(22) says that, if we started with initial conditions such that $\|x_0\|_2 \leq 1$, then, the energy of the respective output

$\|y(t)\|_{L_2}$ scales with the eigenvalues of Y_o. Hence, the observability gramian tells how observable a given initial condition or state is. To see this more clearly, we define observability ellipsoid as the set

$$E_o = \left(Y_o^{\frac{1}{2}} x_0 : x_0 \in C^n \ and \ \|x_0\|_2 = 1\right) \tag{23}$$

This is natural since $\|y(t)\|_{L_2} = x_0^* Y_o x_0$. Let $\mu_1 \geq \mu_2.. \geq \mu_n \geq 0$ be the eigen values and $\phi_1 \geq \phi_2.. \geq ....\phi_n$ be the orthonormal eigen vectors of $Y_o^{\frac{1}{2}}$. Now the orthonormal eigen vectors of the controllability ellipsoid form the principal axis of the ellipsoid and they form an orthonormal basis of the system state space. The eigen values essentially tell, the maximum energy one can get by starting in a certain initial condition ϕ_r such that $\|\phi_r\|_2 = 1$ is μ_r^2. Hence, $\mu_r > \mu_s$ means that ϕ_r is more easily observable than ϕ_s. In conclusion, observaability gramian carries the information about the observable states.

3.3 Hankel Operator

In the previous sections, we divided the input-output representation into input representation with no output and output representation with no input, and tried to understand each of their characteristics. In this subsection, we would like to understand the whole system (1) from both the controllability and observability point of view and understand how important a given state is in the input-output characteristics of the flow. The answer, obviously, lies in the composition of maps from past input to initial conditions (O_c) and initial conditions to the future output (O_o).

$$O_o O_c : \ L_2(-\infty, 0] \rightarrow C^n \rightarrow L_2[0, \infty) \tag{24}$$

This new operator is called Hankel opearator $H = O_o O_c$. This can be viewed as a map from the past input to the future output. Hence, Hankel operator carries information about both the controllability and observability operator and modes. The singular values of the Hankel operator are called the Hankel singular values. The relative importance of a state in the input-output behavior of the flow is given by the corresponding Hankel singular value. Therefore, a steep falling of Hankel singular values implies that, only few states are important in the input-output behavior of flow.

We will show later that, for Navier-Stokes equations linearized about Couette flow, the Hankel singular values drop very steeply.

3.4 Balanced Truncation

As we have seen before, the eigenvalues of the controllability gramain ($X_c^{\frac{1}{2}}$) tell about the relative importance of the controllable modes and eigenvalues of the observabilityy gramain ($Y_o^{\frac{1}{2}}$) tell about the relative importance of the observable modes. In many situations it is possible that, the most controllable modes need not be the most observable modes and vice-versa Therefore, it is not a good idea to delete the weekly controllable modes as they might be the most observable modes and vice versa. This problem can be avoided, if by some means, we can align the controllability and observability ellipsoids perfectly. Then, the weakly controllable modes are also weakly observable modes. It's not at all obvious, if such a transformation exists. It has been shown in [8] that such a transformation (T) exists and is given by

$$T^{-1} = X_c^{\frac{1}{2}} U \Sigma^{-\frac{1}{2}} \tag{25}$$

$$\bar{X}_c = T X_c T^* = \bar{Y}_o = (T^*)^{-1} Y_o T^{-1} = \Sigma \tag{26}$$

Where, U and Σ are given by the singular value decomposition $X_c^{\frac{1}{2}} Y_o X_c^{-\frac{1}{2}} = U\Sigma^2 U^*$. It is now safe to truncate the weak states in this new co-ordinate system.

The error made in the approximation is given by [3]

$$\|G(s) - G_r(s)\|_{H_\infty} \leq 2(\sigma_1^t + ... + \sigma_r^t) \tag{27}$$

where, σ_p^t are the distinct Hankel singular values corresponding to the truncated states.

4 Computations

The computations are done on the stream-wise constant Navier-Stokes equations linearized about Couette flow ($\bar{U} = \frac{1+y}{2}$). After some manipulations they can be written as follows

$$\frac{\partial \psi}{\partial t} = \frac{1}{R} \Delta^{-1} \Delta^2 \psi \tag{28}$$

$$\frac{\partial u}{\partial t} = -\frac{d\bar{U}}{dy} \frac{\partial \psi}{\partial z} + \frac{1}{R} \Delta u \tag{29}$$

Here, Δ is the Laplacian in y and z variables, R is the Reynolds number and ψ is the stream function defined by

$$v = \frac{\partial \psi}{\partial z}, \quad w = -\frac{\partial \psi}{\partial y} \tag{30}$$

The above equations are subject to the no slip boundary conditions on the solid walls

$$\frac{\partial \psi}{\partial y}(y = \pm 1, z, t) = \frac{\partial \psi}{\partial z}(y = \pm 1, z, t) = u(y = \pm 1, z, t) = 0 \tag{31}$$

Taking the Fourier transforms of (28,29) in the z direction, we get

$$\frac{\partial \hat{\psi}}{\partial t} = \frac{1}{R}(D^2 - \alpha^2)^{-1}(D^2 - \alpha^2)^2 \hat{\psi} \tag{32}$$

$$\frac{\partial \hat{u}}{\partial t} = -\frac{i\alpha}{2} \hat{\psi} + \frac{1}{R}(D^2 - \alpha^2) \hat{u}$$

The boundary conditions now are

$$\hat{\psi}(y=\pm 1,t) = D\hat{\psi}(y=\pm 1,t) = \hat{u}(y=\pm 1,t) = 0 \tag{33}$$

where $D = \frac{\partial}{\partial y}$. We now use a finite dimensional approximation of these infinite dimensional equations using Chebyshev collocation in the wall normal direction with Chebyshev-Gauss-Lobatto points. The derivatives are approximated by matrix derivatives with Lagrange interpolation. The normalized kinetic energy per mode is defined as

$$E(t,\alpha) = \frac{\alpha}{16\pi}\int_{-1}^{1} dy \int_{0}^{\frac{2\pi}{\alpha}} (u^2+v^2+w^2)\ dz \tag{34}$$

The discrete version of the above equation is

$$E(t,\alpha) = \frac{1\pi}{8N}\sum_{k=1}^{N-1}[\frac{\hat{u}_k^*\hat{u}_k + \alpha^2\hat{\psi}_k^*\hat{\psi}_k + (D^N\hat{\psi})_k^*(D^N\hat{\psi})_k}{w_k}] \tag{35}$$

Here w is the Chebyshev weight and the subscript, k, denotes the value at the collaction point y_k. C is chosen such that the Eucledian 2 norm of y is the energy and B to chosen be I, the identity operator. After this we can write (28) in the form of (1). We refer the reader to [1] for more details.

5 Results

The results are presented for computations done at R=1000 and $\alpha = 1$. Figure(1) shows the plot of Hankel singular values for N=512, N=256, N=128 and N=64 on a log-log plot. Where, N is the number of collacation points in y direction. The dimensions of n, m, and k in section 2 are n=2N-2, m=2N-2 and k=3N-3 respectively. One can see from the plot that there is a steep falling of Hankel singular values. Hence, only few of the states are important in the input-output properties of the fluid. The plot is also indicating that the right most singular values are sensitive to truncation error and they move a lot on increasing the resolution. The Hankel singular values on the left are very stable and accurate. Figure(2) shows variation of energy, $E(t)$, with respect to time for N=256 and N=128. The initial conditions for these simulations are chosen to be zero and the input is chosen to be $u(t) = u_0\delta(t)$. Where δ is the dirac delta function. It can be seen from the plot that the energy has converged. In the next two plots are plotted the energy verses time of full and truncated models. Figure(3) shows the plot of energy verses time of full model (N=256) and truncated models with 2.5% and 1.7% modes retained. The agreement is pretty good. In Figure(4) are plotted the energy verses time of full model (N=256) and truncated model with 0.8% and 0.4% modes retained. The plot indicates that the agreement is still good between the full model and the truncated model with 0.8% modes retained. There is only a slight discrepancy in the plots near the peak of the energy. The truncated model with 0.4% modes retained, though captures the peak approximately, is performing badly.

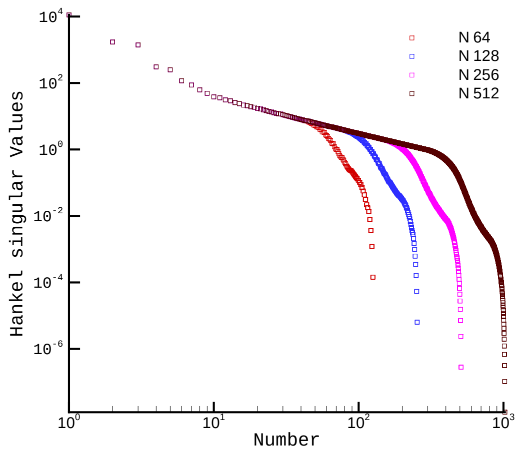

Figure 1. *Hankel singular values*

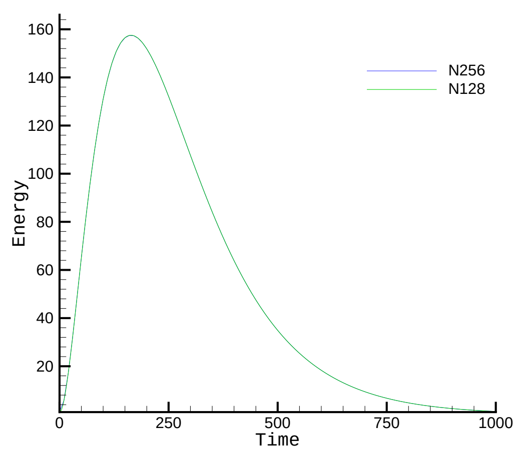

Figure 2. *Energy growth*

6 Conclusions

In this paper, we introduced a novel technique for getting simple models of unsteady fluid phenomena. The main idea behind this method is deleting the weakly controllable and weakly observable states of the flow after the controllability and the observability gramians of the fluid are aligned through a similarity transformation. Computations done on Couette flow using spectral methods indicate that the

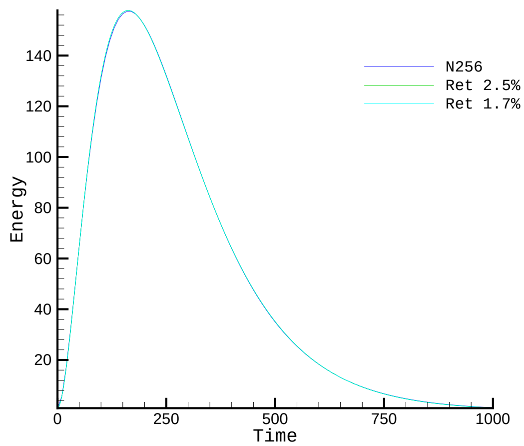

Figure 3. *Energy growth of full and truncated model*

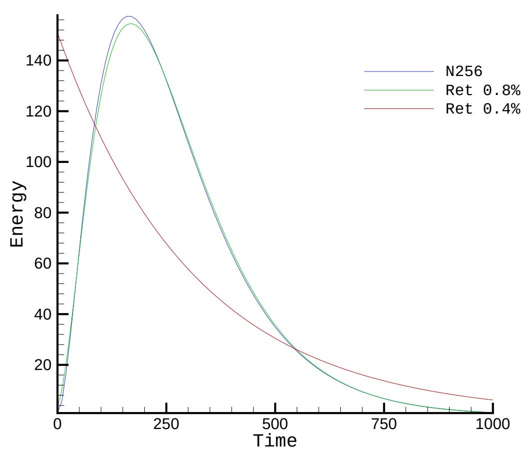

Figure 4. *Energy growth of full and truncated model*

method is performing very well, even for partial differential equations.

7 Future Work

A more natural approach for complexity reduction would be to start discrete time and space equations. This work will be reported else where.

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