Inverse eigenvalue problems involving multiple spectra

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Abstract

If $A \in M_n$, its spectrum is denoted by $\sigma(A)$. If A is oscillatory (O) then $\sigma(A)$ is positive and discrete, the submatrix $A[r+1,\ldots,n]$ is O and its spectrum is denoted by $\sigma_r(A)$. It is known that there is a unique symmetric tridiagonal O matrix with given, positive, strictly interlacing spectra σ_0, σ_1. It is shown that there is not necessarily a pentadiagonal O matrix with given, positive strictly interlacing spectra $\sigma_0, \sigma_1, \sigma_2$, but that there is a family of such matrices with positive strictly interlacing spectra σ_0, σ_1. The concept of inner total positivity (ITP) is introduced, and it is shown that an ITP matrix may be reduced to ITP band form, or filled in to become TP. These reductions and filling-in procedures are used to construct ITP matrices with given multiple common spectra.

1 Introduction

My interest in inverse eigenvalue problems (IEP) stems from the fact that they appear in inverse vibration problems, see [7]. In these problems the matrices that appear are usually symmetric; in this paper we shall often assume that the matrices are symmetric: $A \in S_n$.

If $A \in S_n$, its eigenvalues are real; we denote its spectrum by $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, where $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. The direct problem of finding $\sigma(A)$ from A is well understood. At first sight it appears that inverse eigenvalue problems are trivial: every $A \in S_n$ with spectrum $\sigma(A)$ has the form $Q^T \wedge Q$ where Q is orthogonal and $\wedge = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$. That they are not trivial arises from the fact that A is usually required to have a specified form determined by the underlying graph $\mathcal{G}$ of the system to which it relates.

Almost all of the early, nice, results on IEP relate to the simplest graph, the unbroken *path* on n vertices. For this graph the underlying matrix is tridiagonal with non-zero codiagonal. Physical considerations usually dictate that the matrix is symmetric, that the codiagonal terms are all positive (negative), and that the matrix is positive definite (PD).

As this example shows, there are two principal factors determining the form of the underlying system matrix A. The edge set $\mathcal{E}$ of the graph $\mathcal{G}$ determines which entries in A are, or are not, zero: $a_{ij} \neq 0$ iff $(i,j) \in \mathcal{E}$. There are sign, or positivity, conditions: certain entries or minors of A have specified signs.

There is now a well established classification relating to positivity; for our purposes, the simplest classification is as follows. The matrix $A \in M_n$ is said to be totally positive (TP) (non-negative (TN)) if all its minors are positive (non-negative). It is NTN if it is non-singular (invertible) and TN. It is oscillatory (O) if it is TN and a power, A^m, is TP. It is known that it is O iff it is NTN and has positive codiagonals, i.e., $a_{i,i+1}, a_{i+1,i} > 0, i = 1, 2, \ldots, n-1$.

Perron's theorem on positive matrices, and the Cauchy-Binet theorem together show that if A is TP, then $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, where $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$; this holds also if A is O. Note this holds for all such $A \in M_n$, not just for $A \in S_n$.

Following Ando [1] we let $Q_{k,n}$denote the set of strictly increasing sequences α, of k integers $\alpha_1, \alpha_2, \ldots, \alpha_k$ taken from $1, 2, \ldots, n$. We denote the submatrix of A in rows indexed by α and columns by β, by $A[\alpha|\beta]$, and write $A[\alpha|\alpha] = A[\alpha]$. We use $A(\alpha;\beta)$ for Ando's $\det[A(\alpha|\beta)]$. We define $d(\alpha) = \sum_{i=1}^{k-1}(\alpha_{i+1} - (\alpha_i + 1))$, and denote the subset of $Q_{k,n}$ with $d(\alpha) = 0$ by $Q^0_{k,n}$.

We denote the spectrum of $A[r+1, \ldots, n]$ by $\sigma_r(A), r = 1, \ldots, n-1$, and write $\sigma(A) = \sigma_0(A)$.

With this classification we may identify a symmetric PD tridiagonal matrix with positive (negative) codiagonal as O. If A is O, then so is $A[r+1, \ldots, n]$. This means that $\sigma_r(A)$ consists of $n - r$ discrete positive eigenvalues, and it may be shown that adjacent spectra strictly interlace. In particular, $\sigma_1(A) = \{\mu_1, \mu_2, \ldots, \mu_{n-1}\}$ where

$$0 < \lambda_1 < \mu_1 < \cdots < \lambda_{n-1} < \mu_{n-1} < \lambda_n. \tag{1}$$

The classical result (Gantmacher and Krein [5]) is that there is a unique O tridiagonal matrix $A \in S_n$ with spectra $\sigma(A), \sigma_1(A)$ satisfying (1); there are various efficient ways of finding A, see [2].

2 Isospectral problems

For $A \in S_n$, the isospectral problem is this: given $A^0 \in S_n$ with $\sigma(A^0) = \sigma^0$, find all $A \in S_n$ such that $\sigma(A) = \sigma^0$. Again there is a trivial answer, $A = Q^T A^0 Q$. The problem escapes triviality by demanding that A have a specified form.

If A is required to be tridiagonal and O, there are two well-known ways of computing A. Both depend on first computing $\sigma(A^0) = \sigma^0$:

(i) Choose an arbitrary sequence $\{\mu_i\}_1^{n-1}$ satisfying (1), and construct A as before.

(ii) Choose a sequence $\{x_1, x_{2,} \ldots, x_n\}$ such that $x_i > 0, i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} x_i^2 = 1$, to be the first components $x_1^{(i)}$ of the normalized eigenvectors $x^{(i)}$ of A, and compute A by using the Lanczos algorithm (see [2]).

The procedures i) and ii) are related by the fact that the $(\mu_i)_1^{n-1}$ are the roots of

$$\sum_{i=1}^{n} \frac{x_i^2}{\lambda - \lambda_i} = 0$$

i.e.,

$$\sum_{i=1}^{n} \frac{x_i^2}{\lambda - \lambda_i} = \prod_{i=1}^{n-1} (\lambda - \mu_i) / \prod_{i=1}^{n} (\lambda - \lambda_i).$$

There are two other, less well-known, ways of constructing A that have two advantages: they do not require the step of finding $\sigma(A^0)$; they may be generalized to arbitrary NTN, O or TP matrices. Gladwell [8] proved the following result. Suppose $A \in S_n$ has the property P, one of NTN, O and TP. Take μ not an eigenvalue of A, and compute A^* from

$$A - \mu I = QR, \quad A^* - \mu I = RQ \tag{2}$$

where Q is orthogonal and R is upper triangular with *positive* diagonal (this is always possible). Then $\sigma(A^*) = \sigma(A)$ and A has the same property P. Moreover, if A is a band matrix, then A^* has the same band form. Note that since $\sigma(A)$ is a set of n discrete positive numbers, μ can be *any* negative number and *almost any* positive number.

If $A \in S_n$ is tridiagonal and O, then it may be shown that *all* symmetric, tridiagonal and O matrices B such that $\sigma(B) = \sigma(A)$ may be computed by $n-1$ successive uses of (2), with determined $(\mu_i)_1^{n-1}$.

A family of isospectral matrices A may also be constructed in a Toda flow from A; again the Toda flow preserves each of the properties NTN, O and TP, as also the bandwidth. See [9].

3 Generalizations

We repeat that the classical result for tridiagonal matrices $A \in S_n$ is that there is a unique O matrix with two spectra $\sigma_0(A), \sigma_1(A)$ satisfying (1). Boley and Golub [3] generalized the procedure used to construct tridiagonal A to construct $A \in S_n$ with half bandwidth p from p spectra $\sigma_0, \sigma_1, \ldots, \sigma_{p-1}$. However, if A is required to be O then the strict interlacing of adjacent spectra is necessary but not sufficient to ensure A is O, as the following counterexample shows: $n = 3, p = 2, \sigma_0 = \{1, 4, 6\}, \sigma_1 = \{2, 5\}, \sigma_2 = 3$. Clearly $a_{33} = 3, a_{22} = 4, a_{11} = 4, a_{23}^2 = 2$, so that

$$A = \begin{bmatrix} 4 & a_{12} & a_{13} \\ a_{12} & 4 & a_{23} \\ a_{13} & a_{23} & 3 \end{bmatrix}.$$

Using the products of eigenvalues, we find

$$a_{12}^2 + a_{13}^2 = 4$$

$$48 + 2a_{12}a_{23}a_{31} - 4a_{13}^2 - 3a_{12}^2 - 4a_{23}^2 = 24$$

giving

$$2a_{12}a_{23}a_{31} + a_{12}^2 = 0$$

which clearly has no positive solution. In fact, there is an oscillatory solution for A iff $2 < \sigma_2 < 3$ or $4 < \sigma_2 < 5$.

This counterexample points to a number of questions.

Q1: What are the necessary and sufficient conditions for, say, three spectra $\sigma_0, \sigma_1, \sigma_2$ to be the spectra of, say, a symmetric oscillatory pentadiagonal matrix?

Q2: How can we find the family of oscillatory pentadiagonal matrices with two given strictly interlacing spectra σ_0, σ_1?

These questions were encountered initially in the inverse vibration problem for a beam in flexure. (See [6], [10].) The underlying graph is a *strut*, and the governing matrix is symmetric oscillatory and pentadiagonal. Actually the matrix is a particular kind of oscillatory matrix that we call *inner totally positive* (ITP).

Recall that a sequence $\gamma := \{\gamma(1), \gamma(2), \ldots, \gamma(n)\}$ is a *staircase sequence* if it is non-decreasing and satisfies $\gamma(i) \geq i, \quad i = 1, 2, \ldots, n$. Let ρ (for row) and γ (for column) be staircase sequences. A matrix $A \in M_n$ is called a *ρ, γ-staircase matrix* if $a_{ij} = 0$ when $i > \gamma(j)$ and/or $j > \rho(i)$. If $a_{ij} > 0$ when $i \leq \gamma(j)$ and $j \leq \rho(i)$, then A is said to be a *positive* ρ, γ-staircase matrix. Note that if $\gamma(i) = i$, or $\rho(i) = i$ for some i satisfying $1 \leq i \leq n-1$, then A is reducible. We therefore suppose that $\gamma(i), \rho(i) \geq i+1$ for $i = 1, \ldots, n-1$.

In a staircase matrix, a minor $A\,(\alpha; \beta)$ may be identically zero because it has a zero first or last row or column. If all the other, inner, minors of A are positive, we say that A is ITP. If $\alpha, \beta \in Q_{k,n}$, then $A(\alpha; \beta)$ is an *inner minor* if $\alpha_i \leq \gamma(\beta_i), \beta_i \leq \rho(\alpha_i), i = 1, 2, \ldots, k$.

Note that an ITP matrix (with $\gamma(i), \rho(i) \geq i+1$) is O, but an O matrix need not be ITP, as shown by the counterexample

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Note, however, that a tridiagonal O matrix is ITP; all its inner minors are either principal minors, which are positive, off diagonal entries, which are positive, or products of such quantities.

The following results are established in Gladwell [11]:

- The test for $A \in M_n$ to be TP may be extended to ITP: A is ITP iff $A(\alpha; \beta) > 0$ for all inner minors such that $d(\alpha) = 0 = d(\beta)$.

- A pentadiagonal O matrix need not be ITP.
- A pentadiagonal matrix is ITP iff it is TN and its *corner* minors $A(1,2,\ldots,n-r+1;r,r+1,\ldots,n)$, $A(r,r+1,\ldots,n;1,2,\ldots,n-r+1)$, $r=1,2,3$, are positive.
- A pentadiagonal ITP matrix may be factorised as $A = LU$, and $A^* = UL$ is ITP.
- A pentagidagonal ITP matrix may be factorised as

  $$A = D_1 E^T D_2 E^T D_3 E D_4 E D_5$$

  where

  $$E = \begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & & \\ & & & \ddots & 1 \\ & & & & 1 \end{bmatrix}$$

  and $D_i (i = 1,2,\ldots,5)$ are positive diagonal matrices. If A is symmetric then $A = D_1 E^T D_2 E^T D_3 E D_2 E D_1$.
- If $A \in S_n$ in (2) is pentadiagonal and ITP, then so is A^*.
- If $A \in S_n$ is pentadiagonal and ITP, then so is any B derived from A by Toda flow.
- If $A \in M_n$ is ITP then, under certain conditions, it may be reduced to band form by repeated pre- and post-multiplications by a non-symmetric version of a Givens rotation:

  $$B = S^{-1} A S$$

  where

  $$\begin{aligned} S &= diag(I_{p-1}, R, I_{n-p-1}) \qquad (3) \\ R &= \begin{bmatrix} a & -b \\ c & d \end{bmatrix}, \quad R^{-1} = \begin{bmatrix} d & b \\ -c & a \end{bmatrix}, \quad ad + bc = 1. \end{aligned}$$

  The property ITP is maintained at each such step.
- If $A \in M_n$ is ITP then it may be filled in to become a TP matrix by applying Givens rotations in reverse:

  $$B = SAS^{-1}$$

  and this may be done so that the property ITP is maintained at each step. The matrix is filled in from the bottom, up; once it is filled-in, it may *always* be reduced to band form. Both the filling in and reduction preserve σ; if the minimum value of p used in any reduction or filling-in operation is s, then $\sigma_1, \ldots, \sigma_{s-1}$ will be preserved.

Now we change Q2 slightly:

Q2′: How can we construct a family of ITP pentadiagonal matrices with two strictly interlacing spectra $\sigma_0 = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}, \quad \sigma_1 = \{\mu_1, \mu_2, \ldots, \mu_{n-1}\}$ satisfying (1)?

We proceed as follows:

(i) Construct the unique symmetric tridiagonal O (and therefore ITP) matrix A_0 with $\sigma(A_0) = \sigma_0, \quad \sigma_1(A_0) = \sigma_1$.

(ii) Form $A_1 = DA_0D^{-1}$ with arbitrary positive $D = diag(d_1, d_2, \ldots, d_n)$.

(iii) Fill in A_1 to become a TP matrix A_2.

(iv) Reduce A_2 to pentadiagonal form.

Since the operations in iii), iv) all use $p \geq 2$ in (3), the operations preserve σ_0 and σ_1.

The answer to Q1 for a symmetric ITP pentadiagonal matrix may now be found by using the analysis of Gladwell [6]: we use the Cauchy Binet theorem to find the entries in D_1, D_2, D_3 in terms of modal data that may be computed from $\sigma_0, \sigma_1, \sigma_2$.

Many of these results may be generalised to band and staircase matrices, but there are important open questions regarding matrices of a more general type, such as those for which the underlying graph is a tree. See [4].

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