Characteristic polynomials and pseudospectra

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Abstract

In this paper, we study the ε-lemniscate of the characteristic polynomial in relation to the pseudospectrum of the associated matrix. It is natural to investigate this question because these two sets can be seen as generalizations of eigenvalues. The question of numerical determination of the ε-lemniscate raises the problem of computing the characteristic polynomial p. We can express the coefficients of the characteristic polynomial in the power basis: we use a formation of a Krylov sequence. In order to investigate the practical determination of the characteristic polynomial, we propose a detailed study of its backward error.

keywords : pseudospectrum, characteristic polynomial, rounding error, backward error, ε-lemniscate.

1 Introduction

Eigenvalues of a matrix A are theoretically defined as the roots of its characteristic polynomial. Zeros of polynomials are well known examples of problems whose answers may be highly sensitive to perturbation. In [5] or [18], the authors treat this problem as a condition number question. In [14], it was treated as a continuity question : Mosier defined the notion of root neighborhoods of a polynomial p as the connected components of the set of zeros of all polynomials obtained by coefficientwise perturbation of p of size less than ε.

$$Z(p,\varepsilon) = \{z \in \mathbb{C} : \ \exists q \in P_n, \ q(z) = 0 \text{ with } \|p - q\| \leq \varepsilon\},$$

where $\|.\|$ is metric on $\mathcal{P}_n$ which measures the perturbation of the coefficients of p. $\mathcal{P}_n$ is the set of monic polynomials of degree less or equal to n. In [16], this set is called ε-pseudozero set of p. The relevance of such a set to the conditioning of the zerofinding problem was discussed in [14]. The ε-pseudospectrum $\Lambda_\varepsilon(A)$ is another set defined as the set of eigenvalues of matrices $\hat{A} = A + E$ with $\|E\| \leq \varepsilon$. In [16], they compared the ε-pseudozero sets of polynomials p with the pseudospectra $(\Delta_\varepsilon(A))$

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of associated companion matrices. Their numerical experiments showed that $Z(p, \|p\|\varepsilon)$ and $\Lambda_{\|A\|\varepsilon}(A)$ were generally quite close to one another when A was first balanced. They also gave an algebraic characterization of these ε-pseudozero sets in terms of the level curves of a certain function involving the characteristic polynomial. They used the lemniscate $L(p,\varepsilon) = \{z \in \mathbb{C}, |p(z)| \leq \varepsilon\}$ to bound $Z(p,\varepsilon)$ in order to determine it numerically. Toh and Trefethen used all these sets in order to show that it ought to be possible to compute zeros of a polynomial stably via eigenvalues of its balanced associated companion matrix.

Our aim is different. If p is the characteristic polynomial of A, the set $L(p,\varepsilon)$ appears to be a generalization of the notion of eigenvalues. $\Lambda_{\|A\|\varepsilon}(A)$ is also a natural extension of eigenvalues problems (see [17]). We think that it is theoretically interesting to compare these two generalizations. In section 2, we state inclusions between these sets and we confirm these theoretical results with numerical results.

Second, it is well known that the practical determination of the characteristic polynomial is ill-conditionned. The oldest method for expanding the characteristic polynomial of a matrix is the direct evaluation of principal minors. Then a method due to Leverrier (1840) has been rediscovered and somewhat elaborated in the fourties and fifties. These methods are based upon Newton identities relating the coefficients of an equation with sums of powers of the roots. In [12] Householder and Bauer showed that numerous methods rest mathematically upon the formation of a Krylov sequence. In [12], Wilkinson recalls the Krylov method for computing the coefficients of the characteristic polynomial of a matrix A. This method is based on the following property : Let A be in $\mathbb{C}^{n\times n}$ and $u \in \mathbb{R}^n$ such that
$C = [u, \ldots, A^{n-1}u]$ is invertible, If $\alpha = (a_0, \ldots, a_{n-1})^t$ is the solution of the linear system

$$C\alpha = -A^n u$$

then the characteristic polynomial is $p(x) = \displaystyle\sum_{i=0}^{i=n-1} a_i x^i + x^n$.
We would like to gain insight into the factors responsible for the bad accuracy of the computation. In order to do do that, we study the effects of finite precision arithmetic on this numerical method. We obtain a bound for the backward error. Then we derive a bound for the forward error.

Finally, we use the previous results to investigate the idea that the lemniscates $L(p,\varepsilon)$ defined by the computed characteristic polynomial could be or not a tool to localize eigenvalues.

2 Generalization of eigenvalues involving the characteristic polynomial

We begin this section by introducing some notations.

- $\|.\|_2$ denotes the spectral norm: $\|A\|_2 = \rho(A^*A)^{\frac{1}{2}}$, ρ being the spectral radius.
- $\|.\|_F$ denotes the Frobenius norm: $\|A\|_F = (\sum_{i=0}^{i=n} |a_{ij}|^2)^{\frac{1}{2}}$.
- the metric d on $\mathcal{P}_n$ is defined as follows: $d(p, 0) = \|p\| = (\sum_{i=0}^{i=n} |a_i|^2)^{\frac{1}{2}}$.
- Λ denotes the spectrum.
- $\Lambda(A) = \{\lambda_1, \dots, \lambda_k\}$ with $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_k$.
- P_i denotes the spectral projection associated with λ_i.
- m_i denotes λ_i's algebraic multiplicity and l_i its index.
- For all non singular matrix A, its condition number in the spectral norm is defined by $k_2(A) = \|A^{-1}\|_2\|A\|$.

Mosier [14] studied the sensitivity of the roots to coefficientwise perturbation of the coefficients of the associated polynomial. To treat this question, he defined the notion of ε-neighborhood of a polynomial p in $\mathcal{P}_n$ as

$$N(p, \varepsilon) = \{q \in \mathcal{P}_n, d(p, q) \leq \varepsilon\}.$$

We restrict the set $\mathcal{P}_n$: in the following it denotes the set of monic polynomials of degree n. He also defined the notion of root neighborhood of p as the connected components of the following set

$$Z(p, \varepsilon) = \{z \in \mathbb{C} : z \text{ is a zero of } q \text{ for some } q \in N(p, \varepsilon)\}.$$

Toh et Trefethen [16] called it the ε-pseudozero set. In order to study matrices, the traditional tool was the study of eigenvalues. But when the matrix is not hermitian nor normal, the eigenvalues do not give enough information and the study of ε-pseudospectra is useful (see [6], [10], [16] , [17]). For any matrix A whose order is n, its pseudospectrum is defined as

$$\Lambda_\varepsilon(A) = \{z \in \mathbb{C},\ \exists E \in \mathbb{C}^{n \times n}, \|E\|_2 \leq \varepsilon, z \in \Lambda(A + E)\}.$$

In [16] Toh et Trefethen compared pseudospectra of compagnon matrices with the ε-pseudozeros set of their characteristic polynomial . They showed that these two sets were generally quite close. It exists a set whose numerical determination is easier than the computation of $Z(p, \varepsilon)$:

$$L(p, \varepsilon) = \{z \in \mathbb{C} : |p(z)| \leq \varepsilon\}.$$

In [16], the authors called it ε-lemniscate of p.

As this former set is a generalization of the spectrum, it is important to compare it with the pseudospectrum. In this section, we present theoretical relationships between these sets and numerical examples to illustrate them.

We observe that there are two levels in the perturbation analysis of the eigenvalues: There are changes in the coefficients of the characteristic polynomial which correspond to small changes in the matrix and there are changes in the roots of the characteristic polynomial corresponding to changes in its coefficients. Roughly speaking, the pseudospectrum is a tool to measure the magnitude of these two levels. But it seems that $Z(p, \varepsilon)$ and $L(p, \varepsilon)$ reflect only the second level. Therefore we first propose an inclusion's result from $L(p, \varepsilon)$ into the pseudospectrum of A.

Lemma 2.1 *Under the assumption*
(H) A is an irreducible Hessenberg matrix,
if C denotes the matrix $[e_1, Ae_1, \ldots, A^{n-1}e_1]$,
C is nonsingular and $B = C^{-1}AC$ *is the companion matrix associated with the characteristic polynomial of A.*

PROOF : : see [1], [11].

Theorem 2.2 *Under the assumption*
(H) A is an irreducible Hessenberg matrix,
we have

$$L(p, \varepsilon) \subset Z(p, \varepsilon) \subset \Lambda_{k_2(C)\varepsilon}(A)$$

where $C = [e_1, Ae_1, \ldots, A^{n-1}e_1]$ *and* $k_2(C) = \|C\|_2 \|C^{-1}\|_2$.

PROOF : : In [16], it is proved that

$$L(p, \varepsilon) \subset Z(p, \varepsilon) \subset \Lambda_\varepsilon(B) \tag{1}$$

with B the companion matrix associated with p. We have the result with the lemma 2.1. ■

The inclusion is far from being optimal. It can beobserved on the numerical results.

Then we propose an inclusion result between $\Lambda_\varepsilon(A)$ and a ε-leminiscate of p.We present two different relations based on different mathematical technics. The first inclusion is based on a result about eigenvalues conditioning and the second one uses a perturbation result of Eldelman and Murakami [3].

Theorem 2.3 *With*
$M = max\{m_i \; i = 1, \ldots, k\}$,
$m = min\{m_i \; i = 1, \ldots, k\}$,
$r = min\{\frac{m_j}{l_j} \; i = 1, \ldots, k\}$ $(r > 1)$,
$\gamma(A) = max\{|\lambda_i - \lambda_j| i \neq j\}$,
V_j *the Jordan basis associated with* λ_j , *we have*

$$\Lambda_\varepsilon(A) \subset L(p, \varepsilon')$$

where $\varepsilon' = \left(Kmax(\gamma(A)^n, \gamma(A)^{n-M})\right)\varepsilon^r + 0(\varepsilon^r)$ *with*
$K = max\{(l_j k_2(V_j)\|P_j\|_2)^{\frac{m_j}{l_j}} , \; j = 1, \ldots, k\}$

PROOF : If $z \in \Lambda_\varepsilon(A)$, then $\exists E \in \mathbb{C}^{n\times n}$ such that $\|E\|_2 \leq \varepsilon$ and z is in the spectrum of $A + E$. According to a result concerning eigenvalues conditioning, set in [2], if ε is small enough, it exists $i \in \{1, \ldots, k\}$ such that
$|z - \lambda_i| \leq (l_i k_2(V_i)\|P_i\|_2\varepsilon)^{\frac{1}{l_i}} + O(\varepsilon^{1+\frac{1}{l_i}})$.
We have $p(z) = \prod_{j=1}^{k}(z - \lambda_j)^{m_j} = (z - \lambda_i)^{m_i} \prod_{j\neq i}(z - \lambda_j)^{m_j}$.

So that $|p(z)| \leq [(l_i k_2(V_i)\|P_i\|_2\varepsilon)^{\frac{m_i}{l_i}} + O(\varepsilon^{m_i+\frac{m_i}{l_i}})] \left(\prod_{j\neq i}(|z-\lambda_i|+|\lambda_i-\lambda_j|)^{m_j}\right)$.

Hence $|p(z)| \leq \gamma(A)^{n-m_i}\,(l_i k_2(V_i)\|P_i\|_2)^{\frac{m_i}{l_i}}\,\varepsilon^{\frac{m_i}{l_i}} + O(\varepsilon^{m_i+\frac{m_i}{l_i}})$. ■

The second inclusion that we propose is based on a result of Edelman and Murakami [3]. We recall it and we set a proof of it different from the authors one.

Lemma 2.4 *The companion matrix B associated with $p(x) = \sum_{k=0}^{n-1} a_k x^k + x^n$ is given by*

$$B = \begin{pmatrix} 0 & & & -a_0 \\ 1 & \ddots & & \vdots \\ & \ddots & \ddots\;\; 0 & \vdots \\ & & \ddots\;\; 1 & -a_{n-1} \end{pmatrix} \qquad \in \mathbb{C}^{n\times n}$$

If $E = (e_{ij})$ is a perturbation matrix and $\widetilde{p}$ the characteristic polynomial associated with $B+E$,
we have
$\widetilde{p}(x) = \sum_{k=0}^{n-1} d_k x^k + x^n$ *with*

$$d_k \underset{\|E\|\to 0}{\sim} a_k, \tag{2}$$

(3)

$$d_k - a_k \underset{\|E\|\to 0}{\sim} \sum_{m=0}^{k} a_m \sum_{j=k+2}^{n} e_{j,j+m-k-1} - \sum_{n=k+1}^{n-1} a_m \sum_{j=1}^{k+1} e_{j,j+m-k-1} - \sum_{j=1}^{k+1} e_{j,j+m-k-1}.$$

Proof : : $A = (a_{ij})_{i,j}$ denotes a matrix in $\mathbb{C}^{n\times n}$.
Its characteristic polynomial can be expressed as
$p_A(x) = \sum_{k=0}^{n-1} b_k x^k + x^n$ with
$b_k = (-1)^{n-k} \sum_{J\in C_k} \sum_{\sigma\in S_{n-k}(J)} \varepsilon(\sigma) \prod_{i\in J} a_{i,\sigma(i)}$
with $C_k = \{(i_1,\ldots,i_{n-k}),\ i_1 < i_2 \ldots < i_{n-k}\}$
$S_{n-k}(J)$ the set of all permutations of J.
$\varepsilon(\sigma)$ the signature of the permutation σ.
The classical imbeding from S_{n-k} to S_n implies

$$b_k = (-1)^{n-k} \sum_{\sigma\in J_k} \varepsilon(\sigma) \prod_{i\in I_k} a_{i,\sigma(i)}$$

with $J_k = \{\sigma \in S_n$ for which $\exists L_k = (i_1,\ldots,i_k) : \sigma(i) = i\ \forall i \in L_k$ and $\exists I_k = (i_{k+1},\ldots,i_n)\ \sigma(I_k) = I_k\}$
Applying this result to the matrix $B+E$, we have

$\widetilde{p}(x) = \sum_{k=0}^{n-1} d_k x^k + x^n \qquad$ with

$d_k = (-1)^{n-k} \sum_{\sigma \in J_k} \varepsilon(\sigma) \prod_{i \in I_k} (b_{i,\sigma(i)} + e_{i,\sigma(i)})$ with $B = (b_{ij})_{i,j}$ and $E = (e_{ij})_{i,j}$.

Then

$d_k \underset{\|E\| \to 0}{\sim} (-1)^{n-k} \sum_{\sigma \in J'_k} \varepsilon(\sigma) \prod_{i \in I'_k} b_{i,\sigma(i)}$

with $J'_k \subset J_k$ and I'_k such that $\prod_{i \in I'_k} b_{i,\sigma(i)} \neq 0$

B is a companion matrix so that $b_{i,\sigma(i)} \neq 0 \Leftrightarrow \sigma(i) = i-1$ or $\sigma(i) = n$.

this is possible only if

$I_k = \{k+1, \ldots, n\}$ and σ is such that

$\sigma(k+1) = n$

$\sigma(i) = i - 1 \qquad \forall i = k+2, \ldots, n$

Then $\varepsilon(\sigma) = (-1)^{n-k-1}$.

As $b_{in} = -a_{i-1}$ and $b_{i,i-1} = 1$ we obtain

$d_k \sim (-1)^{n-k}(-1)^{n-k-1}(-a_k) = a_k$. The first property is proved.

We have

$$d_k - a_k \underset{\|E\| \to 0}{\sim} (-1)^{n-k} \sum_{\sigma \in J_k} \varepsilon(\sigma) \sum_{i_0 \in I_k} e_{i_0,\sigma(i_0)} \left(\prod_{i \in I_k i \neq i_0} b_{i,\sigma(i)} \right).$$

Then by determining the sets I_k, J_k and the indices i_0 such that $\prod_{\substack{i \in I_k \\ i \neq i_0}} b_{i,\sigma(i)} \neq 0$, we get the second property. ■

This lemma allows us to state the following theorem.

Theorem 2.5 *Under the assumption*
(H) A is an irreducible Hessenberg matrix,
we have
$\wedge_\varepsilon(A) \subset L(p, \varepsilon")$
with $\varepsilon" = \left(\sqrt{n+1} n^{3/2} k_2(C) \sum_{k=1}^{n} \|p\|^k \right) \varepsilon.$

PROOF : :
It is clear that $\wedge_\varepsilon(A) \subset \wedge_{k_2(C)\varepsilon}(B)$, B being defined in the former lemma and $k_2(C) = \|C\|_2 \|C^{-1}\|_2$.
$z \in \Delta_{k_2(c)\varepsilon}(B) \Rightarrow z \in \Lambda(B + E)$ with $\|E\|_2 \leq k_2(C)\varepsilon$, and according to a former lemma $q(z) = 0$ with $q(z) - p(z) = \sum_{k=0}^{n-1} \delta a_k z^k$ with

$\delta a_k = \sum_{m=0}^{k} a_m \sum_{i=k+2}^{n} e_{i,i+m-k} - \sum_{m=k+1}^{n} a_m \sum_{i=1}^{k+1} e_{i,i+m-k}$ at the first order.

$$\begin{aligned}
\text{Thus } |\delta a_k| &\leq \left(\sum_{m=0}^{n} |a_m| \left(\sum_{i,j} |e_{i,j}|\right)\right) \quad \text{at the first order.}\\
&\leq \left(\sum_{m=0}^{n} |a_m|^2\right)^{1/2} \sqrt{n+1} \left(\sum_{i,j} |e_{ij}|\right)\\
&\leq \|p\|\sqrt{n+1}\; n^{3/2}\|E\|_2\\
&\leq \|p\|\sqrt{n+1}\; n^{3/2}k_2(C)\varepsilon.
\end{aligned}$$

According to [11], as $q(z) = 0$ we have $|z| \leq \|q\|$ so that at the first order $|z| \leq \|p\|$.

$$\begin{aligned}
\text{Thus } |p(z)| = |q(z) - p(z)| &\leq \sum_{k=0}^{n-1} |\delta a_k| \|p\|^k\\
&\leq \|p\|\sqrt{n+1}\; n^{3/2}k_2(C) \sum_{k=0}^{n-1} \|p\|^k \varepsilon\\
&= \sqrt{n+1}\; n^{3/2}k_2(C) \left(\sum_{k=1}^{n} \|p\|^k\right) \varepsilon.
\end{aligned}$$

■

Remark : The factor $k_2(C)$ is not the essential factor in the inclusion's results. If A is an Hessenberg matrix with $a_{i,i-1} = 1\ \forall i$ then it exists a matrix T, such that $k_2(T) = 1$ and $A = TBT^{-1}$, where B is the companion matrix see [4] p 105.

Let us illustrate the relations beetween the sets:

Example: (*Grcar matrix of size 10*). It was introduce by Grcar [8] and was often studied in connexion with pseudospectra. To draw the pseudospectra, we use the pseudospectra GUI, created by Wright and Trefethen [19].

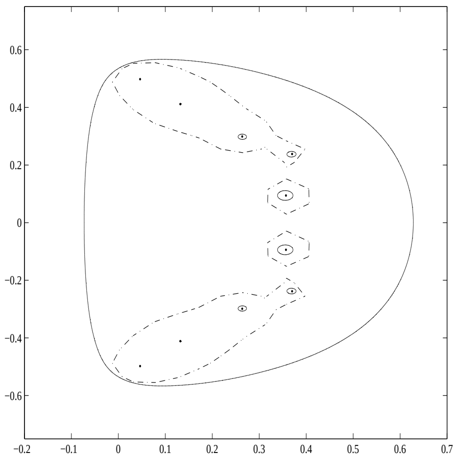

FIGURE 1 :
dashed-dotted curve is the pseudospectra with $\varepsilon = 5.5 \times 10^{-4}$
the solid curves shows $L(p, 5.5 \times 10^{-4})$ and $L(p, \times 10^{-6})$

3 rounding error analysis

The previous section suggests that $L(p, \varepsilon)$ could be a tool to study eigenvalues sensitivity. But we have to know how accurate the determination of this set can be, otherwise it does not give any good informations. Therefore, we are interested in the computational error of the coefficients of the characteristic polynomial. We shall perform a rounding error analysis.

To carry out rounding error analysis of the Krylov method, we need to make some assumptions about the accuracy of the basic arithmetic operations : we work with the standard model of floating point arithmetic [9] :

$$fl(x \; op \; y) = (x \; op \; y)(1+\delta) \quad |\delta| \leq u, \tag{4}$$

where u is the unit roundoff, $op = +, -, *, /$ and $fl(x)$ or $\hat{x}$ denotes the computed value of x.

In the Krylov method applied to an irreducible Hessenberg matrix A of order n, the coefficients $\alpha = (a_0, \dots, a_n)$ of the characteristic polynomial are computed by resolving the linear system

$$C\alpha = b, \quad C = [e_1, \dots, A^{n-1}e_1], \quad b = -A^n e_1 \tag{5}$$

Notations : $a = \|A\|_2$ and $\gamma_m = \dfrac{mu}{1-mu}$.

Theorem 3.1 *Under the assumption* (H), $\hat{\alpha}$, *the computed solution of* $C\alpha = b$ *is the exact solution of the linear system*

$$(C + \Delta C)\hat{\alpha} = b + \Delta b,$$

with

$$\|\Delta C\|_F + \|\Delta b\|_2 \leq \left[(n-1)n^{\frac{3}{2}} a \sqrt{\frac{1-a^{2n-2}}{1-a^2}} + n^{\frac{5}{2}} a^n + n \sqrt{\frac{1-a^{2n}}{1-a^2}} \right] u + O(u^2)$$

PROOF : The rounding error of α is due to the computation of the matrix C and the vector b and to the numerical resolution of the linear system. $fl(C)$ denotes the computed matrix C and $fl(b)$ the computed vector b. According to Higham [9], we have
$\|fl(x_k) - x_k\|_2 \leq \left[(1+\sqrt{n}\gamma_n)^k - 1\right] a^k$, with $x_k = A^k e_1$.

Hence, as $\|fl(C) - C\|_F^2 = \sum_{k=0}^{n-1} \|fl(x_k) - x_k\|_2^2$ and $fl(x_0) = x_0$,

$$\|fl(C) - C\|_F \leq \left[(1+\sqrt{n}\gamma_n)^{n-1} - 1\right] a \sqrt{\frac{1-a^{2n-2}}{1-a^2}}.$$

Then, with the big O notation, we get

$$\|fl(C) - C\|_F \leq a \sqrt{\frac{1-a^{2n-2}}{1-a^2}} \left[(n-1)n^{\frac{3}{2}} u + O(u^2)\right]. \tag{6}$$

and

$$\|fl(b) - b\|_2 \leq a^n \left[n^{\frac{5}{2}} u + O(u^2)\right]. \tag{7}$$

$\hat{\alpha}$ is the computed solution of the linear system $fl(C)w = fl(b)$.
A is an irreducible Hessenberg matrix so that C is upper triangular. So the above linear system will be solved by substitution. According to [9] , $\hat{\alpha}$ satisfies

$$(fl(C) + \Delta T)\hat{\alpha} = fl(b) \tag{8}$$
$$|\Delta T| \leq \gamma_n |fl(C)|. \tag{9}$$

We have then $\Delta C = C - fl(C) + \Delta T$ so that

$$\|\Delta C\|_F \leq \|C - fl(C)\|_F + \|\Delta T\|_F$$

and according to (6), (7) and (8),

$$\|\Delta C\|_F \leq \left((n-1)n^{\frac{3}{2}} a \sqrt{\frac{1-a^{2n-2}}{1-a^2}} \right) u + O(u^2) + n\|C\|_F u + O(u^2)$$

As $\|C\|_F \leq \sqrt{\dfrac{1-a^{2n}}{1-a^2}}$, we get the result. ■

The quantity $\|\Delta C\|_F + \|\Delta b\|_2$ measures the backward error of Krylov method.

Now, with this error analysis result, we are able to derive aa a priori bounds for the forward error of the Krylov method.

Theorem 3.2 *Under the assumption* (H),
If p *is the characteristic polynomial of* A *and* $\hat{p}$ *the characteristic polynomial computed with the Krylov method, We have*

$$\|p - \hat{p}\| \leq \|C^{-1}\|_2$$
$$\left(n^{\frac{5}{2}} a^n + (n-1)n^{\frac{3}{2}} a \sqrt{\frac{1-a^{2n-2}}{1-a^2}} \|\hat{p}\| + n \sqrt{\frac{1-a^{2n}}{1-a^2}} \|\hat{\alpha}\|_2 \right)$$
$$u + O(u^2)$$

PROOF :
α is the solution of $C\alpha = b$ so that

$$C(\alpha - \hat{\alpha}) = -\Delta C \hat{\alpha} + \Delta b.$$

Then

$$\|\alpha - \hat{\alpha}\|_2 \leq \|C^{-1}\|_2 (\|\Delta C\|_2 \|\hat{\alpha}\|_2 + \|\Delta b\|_2).$$

Applying the backward error bound and the property $\|.\|_2 \leq \|.\|_F$, we have the bound. ■

The term $n^{\frac{5}{2}} a^n$, with $a = \|A\|_2$, suggests that if a is greater than 1, the backward error can be disastrous. Thus it can be important to deal with a matrix A whose norm is less than one. This can be achieved by applying a balancing process such as [7], [13], [15].

In the following example, we compute the relative forward error in relation to the norm $\|A\|_2$ and draw its regression line. It confirms that the error is an increasing function of $\|A\|_2$.

$$A = \begin{pmatrix} \frac{1}{8}+\varepsilon^2 & \varepsilon & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{8}\varepsilon-\varepsilon^3 & \frac{1}{4} & \varepsilon & 0 & 0 & 0 & 0 \\ -\frac{1}{8}\varepsilon^2+\varepsilon^4 & \frac{1}{8} & \frac{3}{8} & \varepsilon & 0 & 0 & 0 \\ \frac{1}{8}\varepsilon^3+\varepsilon^5 & -\frac{1}{8}\varepsilon^2 & \frac{1}{8}\varepsilon & \frac{1}{2} & \varepsilon & 0 & 0 \\ -\frac{1}{8}\varepsilon^4-\varepsilon^6 & \frac{1}{8}\varepsilon^3 & -\frac{1}{8}\varepsilon^2 & \frac{1}{8}\varepsilon & \frac{5}{8} & \varepsilon & 0 \\ \frac{1}{8}\varepsilon^5-\varepsilon^7 & -\frac{1}{8}\varepsilon^4 & \frac{1}{8}\varepsilon^3 & -\frac{1}{8}\varepsilon^2 & \frac{1}{8}\varepsilon & \frac{3}{4} & \varepsilon \\ -\frac{1}{8}\varepsilon^6+\varepsilon^8 & \frac{1}{8}\varepsilon^5 & -\frac{1}{8}\varepsilon^4 & \frac{1}{8}\varepsilon^3 & -\frac{1}{8}\varepsilon^2 & \frac{1}{8}\varepsilon & \frac{7}{8}-\varepsilon^2 \end{pmatrix}$$

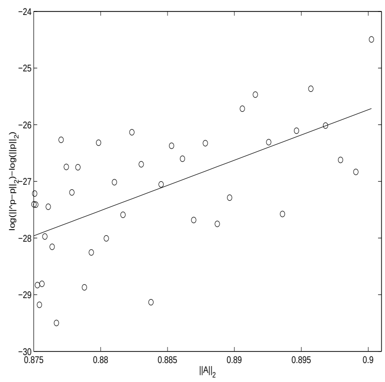

FIGURE 2 : $\frac{\|\hat{p}-p\|_2}{\|p\|_2}$ function of $\|A\|_2$

4 Conclusion

We can rewrite the results of the previous section as follows :

If p is the characteristic polynomial of A and $\hat{p}$ the characteristic polynomial computed by the Krylov method applied to the balanced matrix, then $\hat{p} \in N(p, \eta)$, a neighborhood of p, with
$\eta = \|C^{-1}\|_2 \left(a(n-1)n^{\frac{3}{2}} \|\hat{p}\| + o(n^{\frac{5}{2}}) \right) u + O(u^2)$

In section 2, we set inclusions results between $L(p,\varepsilon)$ and $\Lambda_\varepsilon(A)$. So that $L(p,\varepsilon)$ could be considered as a tool to localize eigenvalues. It can be numerically computed at little cost. But it is well known that the accuracy to compute the characteristic polynomial is limited. So that we can not determine the set $L(p,\varepsilon)$ for small values of ε. At least, ε has to be greater than the error η. Finally that is a limitation that does not exists for the pseudospectra.

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