How to Find Matrix Modifications Keeping Essentially Unaltered a Selected Set of Eigenvalues*

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1 Introduction

In this paper we consider a matrix $A \in \mathbb{C}^{n\times n}$ and we show how to construct a matrix $F \in \mathbb{C}^{n\times n}$ in such a way that $A+F$ keeps practically unaltered a set of ν, $1 \le \nu \le n$, simple eigenvalues of A. The main interest is in the case of matrix modifications F that belong to the subspace generated by the matrices having the same structure of A. However, in this paper, we do not make any restriction on the structure of F. Analogously, no restriction is made in the selection of the eigenvalues.

We start with Wilkinson's theory of the perturbation of a simple eigenvalue $\lambda(A)$ [8], [1] and we characterize unit-norm matrices E that annihilate the coefficient of the first term in the expansion of $\lambda(A+\varepsilon E)-\lambda(A)$ in powers of ε.

Let m denote the number of the entries in E that are not set to zero. We show that if $m \ge \nu+1$ there exist infinitely many matrices E with the required properties and that if $m < \nu+1$ the absence of such matrices E is a generic property. In the latter case, solutions exist under suitable conditions. However, as we shall see, even in case the mentioned conditions are not satisfied, convenient choices of E and of $F = \varepsilon E$ might be found if the tolerance on the invariance of the ν eigenvalues is conveniently weakened.

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Motivations to choose the m entries in A (in E and in $F = \varepsilon E$) to be properly defined might be the following. i) One wants to know how substantially a selected subset of entries in A influences certain eigenvalues. ii) A subset of entries of a given matrix A undergoes to errors heavier than those affecting the remaining ones. iii) The problem that led to consider the matrix A allows entries to range in some sets. iv) Entries are subject to changes caused by modifications to the problem that led to consider the matrix. v) $A = A(\tau_1, \tau_2, \ldots, \tau_p)$ belongs to a family of matrices whose entries depend on parameters $t_1, t_2, \ldots, t_p$ (Homotopies, for instance).

Motivations to choose the ν eigenvalues might be the following. i) They are the largest and/or smallest ones in absolute value. ii) They are the worst conditioned ones [4]. iii) They are clustered. iv) They play an important role in the problem that led to consider the matrix. v) They are the ones closest to the imaginary axis in a spectrum $\sigma(A)$ that lies in the left half of the complex plane.

The outline of the paper is as follows.

In Section 2 we set out the underlying theory. From Wilkinson's perturbation theory we derive a set of equations that characterize the matrices E relevant to the particular case under examination. The set is formed by a homogeneous linear system of ν equations in m unknowns and by a further normalization nonlinear equation. We discuss the possible cases getting the results outlined above.

In Section 3 we outline two algorithms designed to face different situations, discussing in some detail their advantages and their drawbacks.
One of them essentially consists in a procedure for solving the above mentioned set of equations. It works in the case of $m \geq \nu + 1$ and when the homogeneous linear system mentioned before is of maximum rank (i.e. it is a generic homogeneous linear system). It is appropriate for the case of large values of ν.
The other is derived by first transforming the problem represented by the nonlinear system into a direct search optimization one, and then by using the MATLAB functions **mdsmax** and **nmsmax** by N.J. Higham [3] to maximize a suitable function. It is particularly appropriate for the case in which $m \leq \nu$ and the homogeneous linear system mentioned above is of maximum rank (rank= m, no solution), to get useful information, and for the case in which the homogeneous linear system is not of maximum rank - or "close" to one of maximum rank (uncertain maximum rank) - to compute matrices having the properties we require.

Tests and examples carried out using matrices A selected in [3], taken from the literature [2], [5], or randomly generated, are reported in Section 4. Matrices $F = \alpha E$, with α large enough to make the inherent error to dominate the algorithmic one, can be used to get useful information about the algorithmic errors arising from the computation of each considered eigenvalue and about the phenomena arising in case of ill-conditioned eigenvalues [4].

Conclusions are drawn in Section 5.

2 Theoretical results

Let λ be a simple eigenvalue of a matrix $A \in \mathbb{C}^{n \times n}$ and let x and y be respectively the right and left eigenvector associated to λ with $\|x\|_2 = \|y\|_2 = 1$. Following the

classical perturbation theory [8], [1], we can write, for $\varepsilon > 0$ small enough,

$$(A + \varepsilon\, E)\, x(\varepsilon) = \lambda(\varepsilon)\, x(\varepsilon), \qquad \|E\|_2 = 1,$$

where x and λ are differentiable functions such that $\lambda(0) = \lambda$ and $x(0) = x$. Wilkinson proves that one has

$$\left|\frac{d\lambda}{d\varepsilon}\right|_{\varepsilon=0} = \left|\frac{y^H E\, x}{y^H x}\right| \leq \frac{\|y\|_2\, \|E\|_2\, \|x\|_2}{|y^H x|} = \frac{1}{|y^H x|}\,, \tag{1}$$

and points out that the upper-bound is attained if $E = y\, x^H$.

Remark 1. *It is worth noting that in the above outlined theory the Frobenius norm can replace everywhere the* $2-$*norm.*

In the sequel, the matrix $y\, x^H$ will be referred to as the Wilkinson matrix and denoted by W_λ.
The inner product $y^H x$ is usually denoted by $s(\lambda)$,

$$s(\lambda) := y^H x, \tag{2}$$

while $1\,/|s(\lambda)|$ is denoted by $\kappa(\lambda)$ and called the condition number of λ,

$$\kappa(\lambda) := \frac{1}{|y^H x|} = \frac{1}{|s(\lambda)|}\,. \tag{3}$$

In this paper we follow an approach that is the opposite of that outlined above. Instead of considering the matrix E that yields the worst result in (1), we look for a unit-norm matrix E that annihilates the ratio $\left|y^H E\, x\,/ y^H x\right|$.

In order to include any structured and/or sparse modification to A, we assume the matrix E to have at most $m \leq n^2$ nonzero arbitrarily selected entries, that is to have $n^2 - m$ arbitrarily selected zero entries. Also, to investigate the general problem, we consider an arbitrarily chosen set of eigenvalues of A.
This way we look for matrices E such that modifications $\alpha\, E$ to A keep essentially unchanged a set $\widetilde{\sigma}(A) \subseteq \sigma(A)$ of eigenvalues for small enough values of α.

We start by giving the following

Theorem 2. *Let* λ *be a simple eigenvalue of* A*. Let* x *and* y *respectively be the right and left eigenvector associated to* λ*. Let* $\|x\|_2 = \|y\|_2 = 1$*. Let* $E = [e_{ij}]$ *belong to* $\mathbb{C}^{n\times n}$*. One has*

$$y^H E\, x = \sum_{i,j=1:n} \overline{y}_i\, e_{ij}\, x_j = \mathrm{trace}(W_\lambda^H E).$$

Proof. Standard arguments lead to the first equality. The second one easily follows by observing that

$$\sum_{i,j=1:n} \overline{y}_i\, e_{ij}\, x_j = \sum_{i,j=1:n} (\overline{y}_i\, x_j)\, e_{ij} = \sum_{i,j=1:n} (x\, y^H)_{ji}\, e_{ij} = \sum_{j=1:n} (W_\lambda^H E)_{jj}\,.$$

This concludes the proof. □

2.1 Some remarks

Before going any further, it is worth listing some consequences of Theorem 2.

First we observe that if, slightly modifying an idea in [6], we define in the matrix space the inner product

$$(A, B) := \mathrm{trace}(B^H A),$$

that leads to the Frobenius norm, we can read Theorem 2 saying that the matrices E which annihilate $y^H E\, x$ are those orthogonal to the Wilkinson matrix W_λ. We can add that the more E is close to be orthogonal to W_λ, the more $\left|y^H E\, x\right|$ is small.

Then we note that if λ is an ill-conditioned eigenvalue, a matrix E that makes $\left|y^H E\, x \,/ y^H x\right|$ very small can be simply obtained by taking $E = W_\lambda^H$. In fact, W_λ and W_λ^H are unit-norm matrices (both in the 2 and in the Frobenius norm) and

$$\left|\frac{y^H W_\lambda^H\, x}{y^H x}\right| = \left|\frac{y^H\, x\, y^H x}{y^H x}\right| = |s(\lambda)| = \frac{1}{\kappa(\lambda)}\,.$$

It is interesting to note that this agrees with the above results. In fact, a straightforward computation shows that

$$\overline{(W_\lambda, W_\lambda^H)} = (W_\lambda^H, W_\lambda) = s(\lambda)^2,$$

so that the more the eigenvalue λ is ill-conditioned, the more W_λ and W_λ^H are close to be orthogonal to each other.

Finally, note that $W_\lambda^H = W_\lambda$ (i.e. W_λ is Hermitian) if and only if $|s(\lambda)| = 1$ and that the case of a defective eigenvalue associated with a unique Jordan block ($s(\lambda) = 0$) can be viewed as the limit case of orthogonality between W_λ and W_λ^H.

2.2 Further consequences of Theorem 2: a case study

In Theorem 2 we consider only one eigenvalue λ ($\nu = 1$) and we do not assign in advance a structure to the matrix E ($m = n^2$). In this case, to get $\left|y^H E\, x \,/ y^H x\right| = 0$, we have to solve one linear equation in the n^2 unknowns e_{ij} plus the nonlinear equation $\|E\|_2 = 1$, or $\|E\|_F = 1$ (cf. Remark 1). To clarify the matter that follows, it is worth formally re-writing that equation in the form

$$\sum_{k=1}^{n^2} \alpha_k\, \xi_k = 0,$$

where the ξ_k are the unknowns e_{ij} arranged by columns in a n^2-length vector ξ. We put, for $i, j = 1 : n,\ k = 1 : n^2$,

$$k = (j-1)n + i;\ j = \frac{k + n - i}{n}\,,\ i = k - n(j-1),$$

$$\xi_k = e_{ij},$$

$$\alpha_k = \alpha_k(\lambda) = \frac{\overline{y}_i\, x_j}{y^H x}\,.$$

In the general case $1 \leq \nu \leq n,\ 0 < m \leq n^2$, we have to deal with a homogeneous linear system of ν equations in m unknowns and, again, with the nonlinear equation $\|E\|_2 = 1$, or $\|E\|_F = 1$. Again we can represent that homogeneous linear system in a more usual form, writing

$$\sum_{k=1}^{m} \alpha_{hk}\, \xi_k = 0, \quad h = 1 : \nu, \tag{4}$$

even though things become quite complicated. To arrange the unknowns by columns we have to take the $n^2 - m$ zero entries of E into account and properly modify the relationships among $i,\ j$ and k. We denote by $\zeta(i,j)$ the number of the zero entries in E that precede an e_{ij} belonging to the set of the selected m entries and we write, for $i,j = 1 : n,\ k = 1 : m$,

$$k = (j-1)n + i - \zeta(i,j);\ j = \frac{k + n - i + \zeta(i,j)}{n}\,,\ i = k - n(j-1) + \zeta(i,j),$$

$$\xi_k = e_{ij},$$

$$\alpha_{hk} = \alpha_{hk}(\lambda_h) = \overline{y}_i^{(h)} x_j^{(h)} \Big/ y^{(h)H}\, x^{(h)}\,.$$

Here the superscript (h) points out that the two eigenvectors correspond to λ_h.

The above notations allow us to transform our problem into another one which is essentially a Linear Algebra problem. In fact, they show that a specific vector $\xi = (\xi_1, \xi_2, \ldots \xi_m)$ corresponds to a shaped, structured or full matrix E and viceversa. So they are useful since they lead to prove the following theorem.

Theorem 3. *With regard to the existence of matrices E that satisfy the nonlinear system formed by the linear equations in (4) and by the nonlinear equation $\|E\|_2 = 1$ [and by the nonlinear equation $\|E\|_F = 1$], the following two cases can be distinguished.*

1. *If $m \leq \nu$, a generic property is that no matrix E satisfying the above defined nonlinear system exists. Solutions might exist only under suitable conditions.*
2. *If $m \geq \nu + 1$, a generic property is that the matrices E satisfying the above defined nonlinear system belong to a subspace of dimension $m - \nu$ and to the $2-$norm [Frobenius norm] unit-ball. The mentioned subspace of solutions might have a greater dimension only under suitable conditions.*

Proof. The proof is an immediate consequence of the two following considerations. Since the subset of full rank matrices is open and dense in the whole matrix space, a *generic* linear system (4) will have full rank. As a consequence, by virtue of well known Linear Algebra results, only the trivial solution $\xi = 0$ [$E = 0$] exists in case 1, whereas the solutions ξ [the solutions E] belong to a subspace of dimension $m - \nu$ in case 2. Again, well known conditions of Linear Algebra, not generically satisfied, state when solutions in case 1, or a greater dimensional subspace of solutions in

case 2, may arise. The proof follows then by considering the further nonlinear equation $\|E\|_2 = 1$ [the further nonlinear equation $\|E\|_F = 1$] that forces the possible solutions of the homogeneous linear system to belong to the unit-ball. □

Remark 4. *In the case of $m = \nu + 1$, the infinitely many solutions ξ of a generic linear system (4) give rise to only two opposite real normalized matrix solutions.*

3 Outline of two algorithms

We give a sketch of two algorithms we propose. Both are implemented using MATLAB. The refined versions will be adequately discussed in [4].

Trying a numerical solution of the homogeneous linear system (4), then constructing the matrix E^* corresponding to the computed solution ξ^* of (4), and finally normalizing E^*, seems to be the most natural approach to follow.

This idea leads to an algorithm (Algorithm 2) that works efficiently - even in case of large systems - if (4) is a generic (full rank) system with $m \geq \nu + 1$ (see case 2 in Theorem 3). However, it suffers the case of a non generic system and does not work at all when no solution exists. To this regard note that unit-norm matrices E making the ratios $\left| y^{(h)H} E\, x^{(h)} \,/\, y^{(h)H} x^{(h)} \right|$, $h = 1 : \nu$, small enough, might exist even when there is no solution to (4) (i.e. in the generic case considered in Theorem 3, case 1). They would be unit-norm matrices whose corresponding vectors ξ solve the non-homogeneous linear system

$$\sum_{k=1}^{m} \alpha_{hk}\, \xi_k = tol_h, \quad h = 1 : \nu,$$

for particular choices of vector *tol*. Even though those matrices E would work, it would be practically impossible to find a priori the corresponding vectors *tol* which allow to find them.

As a consequence of these drawbacks of Algorithm 2, and to get an algorithm able to carry out any kind of tests, we decided to adopt first a different strategy that could suit even the above cases (Algorithm 1).

The basic idea was that of using the MATLAB functions **mdsmax** and **nmsmax** by Higham [3]. This way we transform the problem of solving the nonlinear system in a direct search optimization problem. In fact, both those functions attempt to maximize a given real function **fun** prepared by the user. Our function **fun** computes

$$1 \Big/ \sum_{h=1}^{\nu} \left| \frac{y^{(h)H} \widetilde{E}\, x^{(h)}}{y^{(h)H} x^{(h)}} \right|, \tag{5}$$

$\widetilde{E}$ being a unit-norm matrix with the same shape or structure of E.

Both Algorithm 1 and Algorithm 2 return the matrix solution E. They both require the user to enter the matrix A, to select the ν eigenvalues to be preserved and to specify the kind of the matrix E. The eigenvalues of A are computed by the MATLAB function **eig**. More detail follows.

3.1 Algorithm 1

To specify the selected kind of the matrix E, the user is required to enter a string. Currently, eight strings are available: 'trid', 'trizd', 'ubid', 'lbid', 'uhess', 'lhess', 'sparse', 'full'. The first six stand for tridiagonal, tridiagonal with zero diagonal, upper bidiagonal, lower bidiagonal, upper Hessemberg and lower Hessemberg. They correspond to six *structuring* functions. String 'sparse' is to be used in case E has neither to be 'full' nor to have one of the above mentioned structures.

Except for the 'full' case, both our function **fun** and Higham's function **mdsmax** [and Higham's function **nmsmax**] must work on a m-length vector - that stands for vector ξ in Sect. 2.2 - to keep the structure or the shape of E. Then a reshaping procedure is needed. In case of structured matrices, we realized such a procedure by means of the relevant structuring function that constructs a suitable reshaping matrix $B \in \mathbb{R}^{n^2 \times m}$, extending a technique due to Francoise Tisseur [7]. In case of sparse matrices, the user is required to enter the m couples of indices defining the shape of E and the reshaping procedure is simply realized by means of the relationships among i, j and k shown in Sect 2.2.

The user is required to select one of the functions **mdsmax** and **nmsmax**. Both of them call **fun** and interact with it to maximize the function implemented therein. The attempt is achieved iteratively. At each iteration, **fun** computes the implemented function at the current data received by the calling routine. In detail, it derives a unit-norm matrix $\widetilde{E}$ from the data and returns the corresponding value of the function in (5). Checks are carried out by **mdsmax** [by **nmsmax**] to verify whether the iteration has to be stopped or not.

We refer the reader to the leading comment lines in the routines in [3] for further detail.

3.2 Algorithm 2

Arbitrarily chosen values are assigned by the user to $m - \nu$ selected components of the unknown ξ in (4) and the homogeneous system is transformed into a non-homogeneous one. The non-singularity of the resulting coefficient matrix is checked. Since normally the right and left eigenvalues are full vectors, such a non-homogeneous linear system turns out to be generally a full one. Thus, Gaussian elimination seems to be the more advisable numerical method. We used the \ MATLAB function.

An auxiliary matrix F that is initialized to the 0 $n \times n$ matrix is used to

1. allow the user to select the $m - \nu$ components of the unknown ξ to be assigned a value (-1 is assigned to $m - \nu$ properly singled out entries in F);

2. allow the user to mark the ν components of ξ that will act as unknowns (1 is assigned to ν properly singled out entries in F);

3. reshape the whole computed m-length solution ξ^* of (4) (the components of ξ^* appropriately substitute for the -1's and the 1's).

Normalization of the final version of F will give the required approximation to the matrix E.

Note that the matrix F also specifies the kind (structure, shape, etc.) of the matrix E (cf. points 1 and 2).

4 Numerical tests

Both the algorithms outlined in Sect. 3 have been checked under MATLAB 6.0.0.88 (R12). In this section we report on the tests we carried out. To make the test repeatable, the command **rand**('seed',0) has been inserted in the implementation of Algorithm 1 before providing **mdsmax** [**nmsmax**] with the starting guess it requires, and issued before entering any randomly generated matrix.

We collect a few tables below. They summarize the results obtained in some tests we singled out since they illustrate significant aspects. Unfortunately, for lack of space, we can not deal with other interesting questions.

Matrices $F = \alpha\, E$, with α large enough to make the inherent error to dominate the algorithmic one have been used. We have taken $\alpha = 10^{\mu}$ **eps**, **eps** being the MATLAB floating point relative accuracy and μ a non negative integer. Then we observed the relevant induced perturbations $|\lambda_h(A) - \lambda_h(A + \alpha\, E)|$. The values assigned to the integer μ are reported in the first row of each table, while the relevant induced perturbations are drawn up in columns, under the values of μ, often roughly reporting only the negative exponent in their normalized exponential representation. The first column shows the indices of the ν eigenvalues we selected as they appear in the list displayed by **eig**. Sometimes the table reports global information related to the other (non selected) eigenvalues in the last row.

A concise information about the tables follows.

Table 1. A=2*(**rand**(20)-.5). The eight negative real part eigenvalues (four complex conjugate pairs) and 'uhess' structure of E were selected.

Table 2. A=2*(**rand**(15)-.5). The four eigenvalues closest to zero (two of them form a complex conjugate pair) and 'full' structure of E were selected.

Table 3. up=2*(**rand**(29,1)-.5); dg=2*(**rand**(30,1)-.5); lw=2*(**rand**(29,1)-.5); A= **full**(**gallery**('tridiag',lw,dg,up)). The largest and the smallest in absolute value eigenvalues and 'trizd' structure of E were selected.

Table 4. As before, but 'ubid' instead of 'trizd' structure of E was selected.

Table 5. A=**gallery**('clement',30). The two eigenvalues closest to zero and 'trid' structure of E were selected.

Table 6. Du=**diag**(-1./**sqrt**(4*(1:14).^2-1),1); Dd=**diag**([-1 **zeros**(1,14)]); Dl= **diag**(1./**sqrt**(4*(1:14).^2-1),-1); $A = Du + Dd + Dl$. Bessel matrix of dimension 15 - spectrum in the left half of the complex plane - (see e.g. [2], [5]). The two eigenvalues closest to the imaginary axis (a complex conjugate pair) and 'sparse' shape of E (i.e. $E(14, 15)$ and $E(15, 14)$) were selected.

Table 7. As before, but dimension 25 instead of 15 and $E(24, 25)$, $E(25, 24)$ instead of $E(14, 15)$, $E(15, 14)$.

Table 8. A=**rand**(10). The last three eigenvalues and 'sparse' shape of E (i.e. $E(1, 2)$, $E(5, 5)$ and $E(9, 10)$) were selected.

Table 9. A=**rand**(30). All the eigenvalues and 'sparse' shape of E (i.e. the diagonal and $E(30, 1)$) were selected.

Table 10. up=2*(**rand**(99,1)-.5); dg=2*(**rand**(100,1)-.5); lw=2*(**rand**(99,1)-.5); A=**full**(**gallery**('tridiag',lw,dg,up)). All the eigenvalues and 'sparse' shape of E (i.e. the diagonal and $E(100, 1)$) were selected.

μ/λ	0↗9	10	11	12	13	14	15
1,2	-15,-14	7.4e-14	7.5e-12	7.5e-10	7.5e-8	7.5e-6	7.5e-4
12,13	-16,-15	6.8e-14	6.5e-12	6.5e-10	6.5e-8	6.5e-6	6.5e-4
16,17	-16,-15	1.7e-13	1.7e-11	1.7e-9	1.7e-7	1.7e-5	1.7e-3
19,20	-16,-15	2.2e-13	2.2e-11	2.2e-9	2.2e-7	2.2e-5	2.2e-3
others	-14↗-8	-7	-6	-5	-4	-3	-2

Table 1

μ/λ	0↗9	10	11	12	13	14	15
12,13	-16,-15	1.7e-13	1.7e-11	1.7e-9	1.7e-7	1.7e-5	1.8e-3
14	-16,-15	1.3e-14	1.3e-12	1.3e-10	1.3e-8	1.3e-6	1.4e-4
15	-16,-15	2.7e-14	2.7e-12	2.7e-10	2.7e-8	2.7e-6	2.7e-4
others	-14↗-8	-7	-6	-5	-4	-3	-2

Table 2

μ/λ	0↗8	9	10	11	12	13	14
20	-16,-15	4.7e-15	1.9e-13	1.9e-11	1.9e-9	1.9e-7	1.9e-5
24	-17,-16	1.0e-15	7.6e-14	7.6e-12	7.6e-10	7.6e-8	7.6e-6
others	-14↗-9	-9,-8	-8,-7	-7,-6	-6,-5	-5,-4	-4,-3

Table 3

μ/λ	0↗8	9	10	11	12	13	14
20	-16,-15	0	1.8e-14	1.5e-12	1.5e-10	1.5e-8	1.5e-6
24	-17,-16	6.9e-15	6.6e-13	6.6e-11	6.6e-9	6.6e-7	6.7e-5
others	-14↗-9	-9,-8	-8,-7	-7,-6	-6,-5	-5,-4	-4,-3

Table 4

μ/λ	11	12	13	14	15	16	17
26	9.5e-13	1.7e-11	1.7e-9	1.7e-7	1.7e-5	1.7e-3	1.4e-1
27	2.5e-12	2.1e-11	1.7e-9	1.7e-7	1.7e-5	1.7e-3	1.4e-1

Table 5

μ/λ	0↗7	8	9	10	11	12	13	14
1,2	-17,-16	-15	-13	-11	-9	-7	-5	-3

Table 6

μ/λ	0↗8	9	10	11	12	13	14
1,2	–16,-15	-13	-11	-9	-7	-5	-3

Table 7

μ/λ	0↗6	7	8	9	10	11	12	13	14	15
8	–16,-15	-15	-14	-13	-12	-12	-9	-7	-5	-3
9	–15↗-13	-11	-10	-9	-8	-7	-6	-5	-4	-3
10	–17,-16	-16	-16	-15	-13	-11	-9	-7	-5	-3
others	–16↗-10	-10,-9	-9,-8	-8,-7	-7,-6	-6,-5	-5,-4	-4,-3	-3,-2	-2,-1

Table 8

μ/λ	0↗8	9	10	11	12	13	14
1:30	-16,-15,-14	-15,-14	-13,-12	-11,-10	-9,-8	-7,-6	-5,-4

Table 9

μ/λ	$0 \nearrow 16$
1:100	-17,-16,-15,-14

Table 10

Except for Tables 8 and 10, all other tables show a behavior of the inherent error $\lambda_h(A) - \lambda_h(A + \alpha E)$ related to the selected eigenvalues that is typical of the case in which the first term in the expansion in powers of α is extremely small (if not zero). The exceptions observable in Tables 8 and 10 are due to reasons that will be explained below.

The results summarized in Tables 1-5 were obtained using both the algorithms. The results summarized in Tables 6-8 can not be obtained using Algorithm 2. In fact, they refer to cases deliberately constructed to make Algorithm 2 to fail.
In all the cases one has $m = \nu$ (case 1 in Theorem 3).
In the first two cases the coefficient matrix in system (4) is declared "close to singular" and "singular to working precision" respectively, and this makes Algorithm 2 to fail. On the contrary, Algorithm 1 succeeded in finding values of (5) very large (1.9e+016 and 6.4e+015 respectively).
In the last case the coefficient matrix in system (4) is definitely non singular and again this makes Algorithm 2 to fail. Algorithm 1 obtained the value 4.5e+001. Thus, the above mentioned first term in the expansion in powers of α is not small and this explains the anomalous behavior of the second row of Table 8. However, the results, on the whole, are not so bad.

We rather used Algorithm 2 to get the results reported in Table 9 and 10. In fact the function (5) to be maximized by **mdsmax** [by **nmsmax**] requires a big amount of computations when ν, n, m are large. Note the very satisfactory result reported in Table 10 that gives an explanation for the exception mentioned above.

5 Conclusions

Given any matrix $A \in \mathbb{C}^{n\times n}$, we show how to determine matrices E (directions in $\mathbb{C}^{n\times n}$) in such a way that $A + \alpha E$ keeps essentially unaltered a set of eigenvalues of A for reasonably large values of α. We show that the problem has solutions, no matter how the set of eigenvalues has been singled out and the kind of E (shape or structure, fullness, etc.) has been selected, provided that a number of entries in E greater than that of the chosen eigenvalues can be properly determined. The opposite case is also discussed in Theorem 3, which gives a complete answer to the existence of such matrices E.

Two algorithms having complementary qualities are briefly described.

Selected tests among the many we carried out are reported. They illustrate interesting aspects and show that satisfactory results can be obtained even when no solution exists according to Theorem 3.

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