D-stability from a numerical point of view

R. Pavani *

1 Introduction

A matrix A is called (positive) D-stable if DA is positive stable, i.e. all eigenvalues have strictly positive real parts, for all positive diagonal matrices D. The concept of D-stability of a matrix was first introduced in mathematical economics [1].

An "ecological" example of D-stability is provided by a community described by Lotka-Volterra population equations. Then, for the matrix of interaction coefficients, the D-stability implies that the stability of the positive equilibrium is preserved under any variation of the equilibrium values of population size. As this concept revealed useful in large-scale systems and in multiparameter singular perturbations, a lot of literature studied this problem (see, for example, [6] and references herein), but an effective method for characterizing D-stability still lacks. For matrices of the second and third orders, constructive necessary and sufficient conditions are known [5]. For the fourth order a constructive stability criterion by the well-known Routh-Hurwitz theorem is presented in both [4] and [7], from different points of view; however practical applications of both criteria are too costly for matrices of higher order. Recently in [6] an amount of thirteenth sufficient conditions, involving a number of important matrices, is presented. However until now the D-stability property looks not to be investigated in its numerical consequences. Here we choose this unusual numerical approach.

Indeed from a numerical point of view a main question arises: let A be D-stable, **what happens when ill-conditioning occurs?** This paper is devoted to answer this question.

*Dipartimento di Matematica - Politecnico di Milano - Piazza L. da Vinci 32 - 20133 Milano-ITALY; rafpav@mate.polimi.it

2 Ill-conditioning for D-stable matrices

In order to study how ill-conditioning affects the D-stability property of a given matrix, at first we recall some well-established theoretical definitions about D-stability and then we present some examples which show how the theoretical D-stability property is not preserved numerically when ill-conditioning occurs in some way.

Definition 1. *Let $M_n(\mathbb{R})$ be the set of square $n \times n$ matrices over the field $\mathbb{R}$ of real numbers, and let $\sigma(A)$ denote the spectrum of a matrix $A \in M_n(\mathbb{R})$. The matrix A is said to be* (positive) stable *if Re $\lambda > 0$ for any $\lambda \in \sigma(A)$.*

Let D_n be the set of $n \times n$ diagonal matrices with positive entries on the main diagonal.

Definition 2. *A matrix $A \in M_n(\mathbb{R})$ is said to be* D-stable *if the product DA is stable for all $D \in D_n$.*

That a D-stable matrix is also stable is easily shown if $D = I$, the $n \times n$ identity matrix.

In the concept of D-stability three different matrices are involved: the given matrix A, the generic diagonal matrix D and their product DA. It is easy to see that ill-conditioning for eigenvalue problem can occur to both A and DA. Obviously any diagonal matrix D is well-conditioned for the eigenvalue problem, however it could present elements dangerously close to zero; so we have to take even this problem under consideration. In the following, we present some significant examples of D-stable matrices where numerical problems appear. The first two examples refer to ill-conditioning, respectively for matrix A and matrix DA; the third one refers to the numerical behavior of matrix D alone. Here we do not go into details about the meaning of ill-conditioning for the eigenvalue problem; about it we suggest [2] and [9].
All the computation were carried out by MATLAB, which exhibits a machine precision of the order of 10^{-16}.
In the following we denote $d_1, d_2, d_3, \ldots, d_n, i = 1, \ldots, n$, the nonnull elements of the diagonal matrix D of order $n \times n$.

Example 1. Let us consider the following matrix

$$A = \begin{bmatrix} 1 & 0 & -a & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{1}$$

where a is a real parameter. This is a well known matrix and in [4] it is proved that A is D-stable for $a \geq -1$.

But what about the conditioning?
For $a = 10^{-14}$ the maximum eigenvalue condition number is $1E + 14$ and the

condition number of the eigenvector matrix is $2.2E + 14$: this means that for A the eigenvalue problem is very ill-conditioned. Then if we choose $d_1 = 1E + 12$, $d_2 = 2E + 5$, $d_3 = 3E - 9$, $d_4 = 4E - 1$, we obtain for the eigenvalues of DA the following values:

$$\lambda_1 = 4E - 1, \quad \lambda_2 = 2E + 1, \quad \lambda_{\mathbf{3}} = \mathbf{0}, \quad \lambda_4 = 1E + 12.$$

We emphasize that for DA the eigenvalue problem is well-conditioned (i.e. for DA the maximum eigenvalue condition number is 1.4142).
We remark that the third eigenvalue is not an approximated value.

Therefore we conclude that the matrix A is **not** numerically D-stable within the used precision because the real parts of the eigenvalues of DA are **not** strictly positive for all positive diagonal matrices D. Indeed the fact that one of the eigenvalues is equal to zero is in concordance with the fact that the matrix DA is singular within the used precision (its condition number is $4.7E + 20$).

Example 2. We consider the same matrix as in Example 1, but another value for a, that is $a = 2$.
In this case the maximum eigenvalue condition number is 1.5, therefore for A the eigenvalue problem is well-conditioned. Then if among all the possible values, we choose $d_1 = 1.5E+12$, $d_2 = 25$, $d_3 = 1.5E-9$, $d_4 = 5$, we obtain for the eigenvalues of DA the following values:

$$\lambda_1 = 5, \quad \lambda_2 = 25, \quad \lambda_3 = 1.5E + 12, \quad \lambda_{\mathbf{4}} = \mathbf{-1.2E - 4}.$$

For DA the eigenvalue problem is now quite ill-conditioned; indeed for DA the maximum eigenvalue condition number is $5.9E+4$ and the condition number of the eigenvector matrix is $1.2E + 5$. This means that there are nearly dependent eigenvectors; therefore numerical problems can arise in solving the eigenvalue problem. The matrix DA is singular within used precision (its condition number is $7E+21$); therefore we expect at least one null eigenvalue within the used precision but we do not find it exactly as the eigenvalue problem is not well-conditioned. Therefore we have to consider the fourth eigenvalue equal to zero with three correct digits only.

As a counterexample, let us maintain the same matrix A and choose $d_1 = 9.3E + 2$, $d_2 = 9.1E + 2$, $d_3 = 4.1E + 1$, $d_4 = 8.9E + 2$. Then the matrix DA is not singular and for DA the maximum eigenvalue condition number is 2.6; therefore no ill-conditioning occurs. As expected all the eigenvalues of DA have strictly positive real parts.

Again we conclude that the matrix A is **not** numerically D-stable within the used precision because the real parts of the eigenvalues of DA are **not** strictly positive for all positive diagonal matrices D.

We remark that *when DA becomes numerically singular the property of D-stability for A is not preserved any longer.* We have shown that sometimes ill-conditioning makes matrix DA to be numerically singular, so in practice ill-conditioning prevents matrix A to be numerically $D-$stable.
However ill-conditioning for matrix A and matrix DA are substantially different: the first one can be checked very easily, whereas the second one cannot be verified

in a finite number of steps; moreover it *cannot be numerically excluded* neither for well-conditioned matrices which are theoretically proved to be D-stable, like matrix A in this Example 2.

Example 3. Let us consider A the Pascal matrix 9×9; it is symmetric and positive definite, therefore it is $D-$stable [3].

We built the Pascal matrix using the command **pascal** by MATLAB so that the matrix presents the first row with elements equal to 1, the second row with elements that are integers from 1 to 9, and so on.
Then if among all the possible values, we choose $d_1 = 8.6E-14$, $d_2 = 5.7E-14$, $d_3 = 9.8E-14$, $d_4 = 7.9E-14$, $d_5 = 1.5E-14$ $d_6 = 8.3E-14$, $d_7 = 1.9E-14$, $d_8 = 6.4E-14$, $d_9 = 6.7E-14$, we obtain for the eigenvalues of DA the following values:

$$\begin{aligned}
&\lambda_1 = 1E-9, \quad \lambda_2 = 2.4E-11, \quad \lambda_3 = 1.9E-12, \ldots \lambda_4 = 3.1E-13,\\
&\lambda_5 = 7.4E-14, \quad \lambda_6 = 1.3E-14, \quad \lambda_7 = 1.2E-15, \ldots\\
&\boldsymbol{\lambda_8 = 1.2E-16}, \quad \boldsymbol{\lambda_9 = 1.7E-18}.
\end{aligned}$$

In this case both A and DA are well-conditioned for the eigenvalue problem, as the maximum condition number is, respectively, 1 and 1.3; moreover matrix DA is far from singularity (its condition number is $8.2E+8$).
Nevertheless, from a numerical point of view, λ_8 and λ_9 have to be considered equal to zero within the used precision; therefore again it seems that matrix A cannot be considered numerically $D-$stable.
However in this case the numerical behavior is not due to any ill-conditioning, but it can be explained by the fact that the elements $d_i, i = 1, 2, \ldots, 9$, are all of the order of 10^{-14}, very close to the used machine precision; therefore a kind of dangerous rescaling happens and this is the reason why some eigenvalues numerically annihilate.

Indeed if we use for d_i the same numerical values, but with a different order (say 10^{-12}), all the eigenvalues exhibit strictly positive real parts, as expected.

To study better this case we can resort to the matrix *sign of* A, denoted as $sgn(A)$. For any matrix A with no eigenvalues lying on the imaginary axis in the complex plane, $sgn(A)$ is defined as the diagonal matrix whose elements are as many $+1$ as eigenvalues of A with positive real parts and as many -1 as eigenvalues of A with negative real parts. It is proved (e.g. [8]) that the sequence defined as follows:

$$\begin{aligned}
A_1 &= A\\
A_{k+1} &= (A_k + A_k^{-1})/2 \qquad k = 1, 2, \ldots
\end{aligned}$$

converges (quickly) to $sgn(A)$.

For the matrix DA considered in this example, we built the sequence A_{k+1}, with $A_1 = DA$; actually for $k = 80$, we obtained A_{k+1} as a diagonal matrix with all nonnull elements equal to 1; this means that the matrix A has to be considered a numerically D-stable matrix.

3 Conclusion

We have presented some significant examples which enlighten discordance between the numerical behavior and the expected theoretical one for some matrices, whose D-stability is theoretically proved.

These events cannot be disregarded; so we propose that a slightly different definition of D-stability is accepted from a numerical point of view.

A matrix $A \in M_n(\mathbb{R})$ is said to be numerically D-stable if it is well-conditioned for the eigenvalue problem and the product DA is stable for all $D \in D_n$, which do not make DA ill-conditioned for the eigenvalue problem.

Using this definition of numerical D-stability, we can recover Example 2 (and, obviously, Example 3) and state that those matrices are not only theoretically but also numerically D-stable; on the contrary matrices of the kind of that one presented in Example 1 are to be considered D-stable just from a theoretical point of view, but not from a numerical point of view. We remark that even using this proposed definition of numerical D-stability, the set of numerical D-stable matrices reveals a subset of the set of the D-stable matrices theoretically defined.

Acknowledgement The author is indebted to Lidia Aceto for fruitful discussions and helpful suggestions.

Bibliography

[1] K. J. Arrow and M. McManus, *A Note on Dynamic Stability*, Econometrica, 26 (1958), pp. 448–454.

[2] J. W. Demmel, *Applied Numerical Linear Algebra*, SIAM, 1997.

[3] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, 1991.

[4] R. Johnson and A. Tesi, *On the D-stability Problem for Real Matrices*, Boll. UMI, 8, 2-B, (1999), pp. 299–314.

[5] C. R. Johnson, *Second, Third and Fourth Order D-stability*, J. Res. Nat. Bur. Standards USA, 78, 1, (1974), pp. 11–13.

[6] W. S. Kafri, *Robust D-stability* Appl. Math. Lett., 15 (2002), pp. 7–10.

[7] G. V. Kanovei and D. O. Logofet, *D-Stability of 4-by-4 Matrices*, Computational Mathematics and Mathematical Physics, 38, 9 (1998), pp. 1369–1374.

[8] M. Marcus, *Matrices and Matlab*, Prentice-Hall Inc., 1993.

[9] L. N. Trefethen and D. Bau, *Numerical Linear Algebra*, SIAM, 1997.