# Powers of Ray Pattern Matrices


***Jeffrey L. Stuart***[*]



## Abstract

We examine the similarities and differences between results for powers of sign pattern matrices and for powers of ray pattern matrices. In particular, we investigate what is known about $k$-potent patterns and powerful patterns in both the irreducible and reducible cases.


## 1 Signs, Rays, Sign & Ray Patterns

Since at least the 1960's, the qualitative theory of real matrices has been an extremely fruitful area of research. When posing qualitative questions, we ask what aspects of a matrix — such as stability, invertibility, controllability — are determined entirely by the sign pattern — the arrangement of positive, negative and zero entries — without reference to the magnitudes of the entries. During the last decade, there has been a vigorous effort to develop a corresponding qualitative theory of complex matrices. One of the main approaches has been to recognize that positivity and negativity naturally generalize to rays of fixed argument in the complex plane. That is, extending the view that all positive numbers as equivalent to the number $+1$, and all negative numbers as equivalent to the number $-1$, we will treat the complex ray consisting of all complex numbers of the form $re^{i\theta}$ with $r > 0$ and $\theta \in \mathbb{R}$ as equivalent to $e^{i\theta}$. Of course, we need to introduce an appropriate arithmetic on such rays. Clearly, $r_1e^{i\theta} + r_2e^{i\theta}$ will be equivalent to $e^{i\theta}$ provided $r_1 > 0$ and $r_2 > 0$. In general, however, the choice of $\theta$ for $r_1e^{i\theta_1} + r_2e^{i\theta_2}$ to be equivalent to $e^{i\theta}$ will depend not only on $\theta_1$ and $\theta_2$, but also on $r_1$ and $r_2$. Consequently, we will require an additional symbol, #, to represent the result of adding nonzero complex numbers that lie on distinct rays. A ray operation that results in # will be called *ambiguous*. The complete arithmetic properties of # can be found


[*]Department of Mathematics, Pacific Lutheran University, Tacoma, WA 98447 USA, jeffrey.stuart@plu.edu

in [11] or [6].

If $A$ is a fixed, $m \times n$ real matrix, the qualitative class of $A$, called the *sign pattern of* $A$, is the set of all $m \times n$ real matrices $B$ such that $sign(b_{ij}) = sign(a_{ij})$ for all $i$ and $j$. For convenience, the class is often represented by an appropriate canonical element: the unique member of the class whose entries are in $\{0, -1, +1\}$. An $m \times n$ *generalized sign pattern* is obtained by replacing one or more entries in a canonical representative of a sign pattern by the symbol #. The real matrix $B$ is an element of a generalized sign pattern exactly when $sign(b_{ij})$ agrees with the the sign of each corresponding unambiguous entry of the canonical representative.

If $A$ is a fixed, $m \times n$ complex matrix, the qualitative class of $A$, called the ray *pattern of* $A$, is the set of all $m \times n$ complex matrices $B$ with the same zero-nonzero pattern as $A$ such that $\arg(b_{ij}) = \arg(a_{ij})$ (modulo $2\pi$) whenever $a_{ij} \neq 0$. For convenience, the class is often represented by the unique member of the class whose entries are in $\mathcal{S} = \{z \in \mathbb{C} : |z| \in \{0, 1\}\}$. An $m \times n$ *generalized ray pattern* is obtained by replacing one or more entries in a canonical representative of a ray pattern by the symbol #. That is, by allowing one or more entries in the matrices of the pattern to have unspecified argument (or be zero).

A pattern $B$ is called a *subpattern* of a pattern $A$ if either $B$ equals $A$ or else $B$ can be obtained from $A$ by replacing one or more nonzero entries with a zero. Two special patterns will be used extensively: $I_n$, the $n \times n$ identity, and $J$, the $m \times n$ matrix all of whose entries are 1's.

When working with pattern classes, it is often useful to know what transformations send sign (ray) patterns to sign (ray) patterns. Clearly, scalar multiplication and transposition do so. While matrix multiplication generally does not transform one sign (ray) pattern into another, multiplication by certain patterns does. In particular, pre- or post-multiplication by a pattern whose canonical representative is a permutation matrix either fixes a pattern, or transforms it into another pattern. Thus permutation equivalence and permutation similarity are useful transformations. Diagonal scaling, that is, pre- or post-multiplication by a nonsingular diagonal pattern, sends patterns to patterns. Observe that nonsingular, diagonal sign patterns are self-inverses as patterns; and that nonsingular, diagonal ray patterns are invertible as patterns via conjugation. Thus another useful pattern transformation is diagonal similarity.

## 2 Some Qualitative Questions

Among the many questions concerning sign and ray patterns, we will briefly mention one and then focus on another.

Perhaps the oldest question about sign patterns is which sign patterns are invertible? Work on the invertibility of sign patterns extends back to work by Bassett, Maybee and Quirk ([1]), and includes much other work including [4] and [8]. During the last ten years, attention has shifted to the subject of ray patterns and invertibility, see [7], for example.

The question on which this paper focuses, is what can be said about powers of patterns? Suppose that $A$ is a square matrix, real or complex, certainly $A^2$ is well-defined. However, if $A$ is a sign or ray pattern, what exactly does $A^2$ mean? If $A$ is a square sign or ray pattern, we will denote by $A^2$ the (possibly generalized) sign or ray pattern corresponding to the set of all products of two real or complex matrices from the pattern $A$.

One natural place to begin is, for what patterns is $A^2 = A$, or more generally, $A^{k+1} = A$ for some positive integer $k$? Such patterns are called *sign $k$-potent sign patterns*, or *pattern $k$-potent ray patterns.* Sign $k$-potent patterns were investigated by Eschenbach, Hall, Johnson and Li [3] in the strictly nonzero case with $k = 2$; by Stuart, Eschenbach and Kirkland [12] in the irreducible case for general $k$, and by Stuart [9] in the reducible case. The corresponding results for pattern $k$-potent ray patterns was investigated by Stuart, Beasley and Shader [11] in the irreducible case; and by Stuart [10] in the reducible case. The irreducible patterns have been fully characterized. Necessary conditions (partial characterizations) are known for the reducible patterns.

It is easy to show that if $A^{k+1} = A$ for some positive integer $k$ where $A$ is either a sign pattern or a ray pattern, then in fact $A^h$ is actually a sign pattern or ray pattern for every positive integer $h$. Thus naturally, we are lead to the question: For what sign (ray) patterns is $A^h$ actually a sign (ray) pattern for every positive integer $h$? Such patterns are called *powerful.*

Powerful sign patterns were investigated by Li, Hall and Eschenbach in [5]. They showed that if $A$ is an $n \times n$ powerful pattern, then $A$ is periodic, that is, there existed unique, smallest positive integers $b$ and $p$, called the *base* and *period*, respectively, such that $A^{b+p} = A^b$, and that $p < 3^n$. Notice that when $b = 1$, $A$ is sign $p$-potent. Also, notice that if the sign pattern $A$ has base $b$ and period $p$, then $A^{b+bp} = A^b$. Letting $M = A^b$, it follows that $M^{p+1} = M$, so that $M$ is sign $k$-potent for some $k$ that divides $p$. A complete characterization for the irreducible, powerful sign patterns is given in [5],

The case of powerful ray patterns is more complicated than that of powerful sign patterns. The simple $1x1$ example $A = [e^i]$ is clearly powerful but there exist no positive integers $b$ and $p$ such that $A^{b+p} = A^b$. Li, Hall and Stuart showed in [6] that for the irreducible, powerful ray pattern $A$, there always exists a $\theta \in \mathbb{R}$ such that $B = e^{i\theta}A$ has base and period. Returning to the preceding example, $\theta = -1$, and $B = [1]$; clearly $B$ has base $b = 1$ and period $p = 1$.

If the powerful ray pattern $A$ has base $b$ and period $p$, then $A^{b+bp} = A^b$. Letting $M = A^b$, it follows that $M^{p+1} = M$, so that $M$ is pattern $k$-potent for some $k$ that divides $p$. Consequently, we would expect that the structure of pattern $k$-potent ray patterns is closely related to the structure of powerful ray patterns.

For convenience, we note here that the set of powerful ray patterns is closed under scalar multiplication by a ray, diagonal similarity, permutation similarity, transposition, conjugation, direct sums and the formation of subpatterns.

## 3 Canonical Forms for Irreducible Pattern $k$-Potent Ray Patterns

Suppose that $A$ is a square, irreducible ray pattern. It is well known (see [2, Section 3.4], for example) that there is a unique largest, positive integer $m$, called the *index of imprimitivity of* $A$, such that $A$ is permutation similar to an $m \times m$ block partitioned, ray pattern of the form:

$$\widehat{A} = \begin{bmatrix} 0 & A_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & A_2 & 0 & \dots & 0 \\ 0 & \ddots & 0 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & & \ddots & 0 & A_{m-1} \\ A_m & 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \qquad (*)$$

where the diagonal blocks are square. Further, $\widehat{A}$ is unique up to permutation within the blocks and up to cyclic permutation of the sequence of the blocks. The matrix $\widehat{A}$ given by (*) is called the *cyclic form of* $A$. When $m = 1$, $A$ is its own cyclic form, and it will be understood that $A = \widehat{A} = A_1$.

It should be apparent that if the irreducible, ray pattern $A$ is in cyclic form with index of imprimitivity $m$, and that if $A$ is pattern $k$-potent, then $m$ divides $k$.

Let $P_n$ denote the $n \times n$ circulant permutation matrix with first row $(0, 1, 0, \dots, 0)$. Let $w$ be a primitive $h^{th}$ root of unity for some positive integer $h$. (By convention, $w = 1$ is a primitive $1^{st}$ root of unity.) Then $wP_n$ is invertible as a ray pattern, with inverse ray pattern $(wP_n)^* = \overline{w}P_n^T$. Also note that $n$ is the smallest positive integer such that $P_n^n = I_n$, implying that $(wP_n)^s = I_n$ whenever $s$ is divisible by both $h$ and $n$. In particular, $\ell = \operatorname{lcm}(h, n)$ is the smallest such positive integer. Thus, $wP_n$ is pattern $\ell$-potent.

Every square, generalized ray pattern admits a symmetric block partition such that each block has the form $\alpha J$ where $\alpha \in S \cup \{\#\}$ and $J$ is the all ones matrix of the appropriate size. A coarsest block partition of this type is one that is not a proper subpartition of any other symmetric block partition with blocks of the form $\alpha J$. It was noted in [10] that the coarsest block partition of this type always exists and is unique for generalized ray patterns. For a square, generalized ray pattern $A$, the *reduced block matrix for* $A$, denoted $red(A)$, is the unique ray pattern induced by the coarsest partitioning of $A$. That is, if the blocks of $A$ in a coarsest partition are $\alpha_{hj}J_{hj}$ for $1 \leq h, j \leq m$ for some $m$, then $red(A)$ is the $m \times m$ ray pattern whose $(h, j)$-entry is $\alpha_{hj}$. Observe that $wP_n$ for $w \in \mathcal{S}$ is its own unique, reduced block matrix.

The following results are Theorem 1 and Theorem 2 of [10].

**Theorem 1.** *Let $A$ be a square, generalized ray pattern. Then for each positive integer $k$, $red(A^k) = red([red(A)]^k)$. Further, if $A^{k+1} = A$ for some positive integer $k$, then $[red(A)]^{k+1} = red(A)$.*

**Theorem 2.** *Let $A$ be a nontrivial, irreducible ray pattern. Let $k$ be a positive integer. Then $A$ is pattern $k$-potent if and only if $A$ can be transformed via signature similarity and permutation similarity into a ray pattern $B$ such that $red(B)$ is $wP_m$ where $m$ is some positive integer that divides $k$, and where $w$ is a complex number such that $w^m$ is a primitive $(k/m)^{th}$ root of unity.*

Note that diagonal similarity preserves cyclic form. Thus the previous result can be restated as follows:

**Theorem 3.** *Let $A$ be a nontrivial, irreducible ray pattern with index of imprimitivity $m$. Suppose that $A$ is in cyclic form (*). Then $A$ is pattern $k$-potent if and only if, after an appropriate diagonal similarity, $A_h = wJ_h$ for $1 \leq j \leq m$, where $J_h$ has the same dimensions as $A_h$, and where $w^m$ is a primitive $(k/m)^{th}$ root of unity.*

## 4 Canonical Forms for Irreducible Powerful Ray Patterns

Theorem 3.4 and the remarks preceding Theorem 3.5 of [6] yield:

**Theorem 4.** *Let $A$ be a nontrivial, irreducible ray pattern with index of imprimitivity $m$. Suppose that $A$ is in cyclic form (*). Then $A$ is powerful if and only if, after an appropriate diagonal similarity, there exists a ray $w$ such that $A_h$ is a subpattern of $wJ_h$ for $1 \leq j \leq m$, where $J_h$ has the same dimensions as $A_h$.*

Notice that if $A$ is powerful with index of primitivity $m$, and if $w$ is the ray given in the preceding theorem, then $w^{-1}A$ is either pattern $m$-potent, or else the blocks $w^{-1}A_h$ can be filled in to yield a pattern $m$-potent ray pattern. Consequently, the following result, which is Theorem 3.6 of [6], is not surprising. Apparently raising $B$ to the $b^{th}$ power as given by the condition $B^{b+p} = B^b$ is to fill in the zeros in the blocks $B_h$.

**Theorem 5.** *Let $A$ be an irreducible ray pattern. Then $A$ is powerful if and only if there is a ray $c$ such that $B = cA$ is periodic.*

We close this section with a surprising consequence of the preceding results.

**Corollary 6.** *An irreducible, powerful ray pattern is always a subpattern of an entrywise nonzero, powerful ray pattern.*

## 5 Cyclic Normal Form for Pattern $k$-Potent Ray Patterns

The rest of this paper discusses the powers of reducible ray patterns.

Suppose that $A$ is permutation similar to

$$\left[\begin{array}{cc} 0_{m\times m} & 0_{m\times n} \\ 0_{n\times m} & B \end{array}\right].$$

Note that $A^{k+1} = A$ if and only if $B^{k+1} = B$, and that $A$ is powerful if and only if $B$ is powerful. Consequently, we will ignore the zero rows and zero columns having a common index set. Such rows and columns will be called *extraneous zero rows and columns*. Henceforth, we will assume that all ray patterns under consideration have had all extraneous zero rows and columns removed.

If the ray pattern $A$ is block upper triangular and pattern $k$-potent, then necessarily each of the diagonal blocks $A_{jj}$ must satisfy $A_{jj}^{k+1} = A_{jj}$. Then by Lemma 9 of [11], for each $j$, there exists $k_j$ dividing $k$ such that $A_{jj}$ is pattern $k_j$-potent, and hence $k$ must be divisible by the least common multiple of the $k_j$. If the ray pattern $A$ is block upper triangular and powerful, then necessarily each of the diagonal blocks $A_{jj}$ must also be powerful

Suppose that the ray pattern $A$ has a Frobenius normal form without extraneous zero rows and columns, and with $n$ irreducible, diagonal blocks, $A_{jj}$ such that each of the $A_{jj}$ is pattern $k_j$-potent for some positive integer $k_j$.. If $A_{jj}$ is the $1 \times 1$ zero matrix, use the $1 \times 1$ pattern $I_1$ as a diagonal matrix and a permutation matrix. If $A_{jj}$ is not a zero matrix, then apply Theorem 3 to obtain permutation and diagonal matrices such that $A_{jj}$ is isomorphic to a ray pattern $\widehat{A}_{jj}$ such that $red(\widehat{A}_{jj})$ is of the form $w_j P_{m_j}$ for some positive integer $m_j$ and some complex number $w_j$ such that $w_j^{m_j}$ is a primitive $(k_j/m_j)^{th}$ root of unity. Using the direct sum of the permutation similarities for the diagonal blocks and the direct sum of the diagonal similarities for the diagonal blocks, $A$ is isomorphic to a ray pattern $\widehat{A}$ in Frobenius normal form whose irreducible, diagonal blocks are either $1 \times 1$ zero matrices or else the $\widehat{A}_{jj}$. Any matrix $\widehat{A}$ in this form is called the *cyclic normal form for $A$*.

# 6 The Role of $red(A)$ for Reducible, Pattern $k$-Potent Ray Patterns

It is not the partition induced by the Frobenius normal form of $A$ but rather the subpartition induced by the cyclic form of each irreducible, diagonal block of $A$ that is crucial to understanding the structure of $A$ when $A$ is pattern $k$-potent. The following result, which is Theorem 7 of [10], parallels the results developed for the reducible sign patterns in [9].

**Theorem 7.** *Let $A$ be a ray pattern in cyclic normal form with no extraneous zero rows and columns. Let $A$ be subblock partitioned by the partition induced by the cyclic blocks of each irreducible, diagonal block of the cyclic normal form. The following are equivalent:*

*(i)* *The matrix $A^{k+1} = A$ for some positive integer $k$;*

*(ii)* *For all $h$ and $j$, the subblock $A_{hj} = \alpha_{hj} J_{hj}$ where $\alpha_{hj} \in \mathcal{S}$ and $J_{hj}$ is the all*

*ones matrix the same size as $A_{hj}$; and the matrix of coefficients, $red(A)$, satisfies $[red(A)]^{k+1} = red(A)$ for some positive integer $k$.*

One important consequence of Theorem 7 is that the problem of classifying which ray pattern are pattern $k$-potent reduces to examining those patterns in cyclic normal form for which the irreducible, diagonal blocks are $1 \times 1$ zero matrices and matrices of the type $wP$.

If we focus on $red(A)$ rather than $A$, then necessarily we should like to understand the nature of the off-diagonal blocks of $red(A)$. It turns out that the following lemma from [10] allows us to completely determine the structure of such blocks.

**Lemma 8.** *Let $A$ be a block upper triangular ray pattern with diagonal blocks $A_{jj}$. Suppose $s > r$. If $A^{k+1} = A$ for some positive integer $k$, then $(A_{rr})^{k-h} A_{rs} (A_{ss})^h$ is a subpattern of $A_{rs}$ for $0 \le h \le k$. Further, if $A_{rr}$ is of type $w_r P_{m_r}$ and if $A_{ss}$ is of type $w_s P_{m_s}$, then $A_{rr} A_{rs} = A_{rs} A_{ss}$.*

The complete characterization of the individual, off-diagonal blocks then follows from this lemma and results in [13]. Open questions remain concerning the relationships between parameters in distinct, off-diagonal blocks. See [10] for the details.

# 7 Reducible, Powerful Ray Patterns

At first glance, it would seem that the development of the cyclic normal form for reducible, pattern $k$-potent ray pattern matrices should be immediately adaptable to the case of powerful ray patterns. However, the fact that the blocks $\widehat{A_{jj}}$ in a powerful matrix are *subpatterns* of matrices $B_{jj}$ that can be reduced to smaller matrices $red(B_{jj})$ apparently prevents the direct exploitation of this structure. An alternative approach, suggested by Theorem 5 also runs into difficulties. To see this, consider the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & e^i \end{bmatrix}.$$

Clearly, $A$ is powerful, however, there exists no ray $c$ such that $B = cA$ is periodic. Indeed, there exists no diagonal similarity by any nonsingular diagonal pattern $D$ such that $DAD^*$ is periodic. The diagonal scaling matrix $D = diag(1, e^{-i})$ does satisfy $DA$ is periodic, but unfortunately, diagonal scaling, as opposed to diagonal similarity, is not a useful tool for understanding the structure of powerful matrices.

We have now reached the current state of development with respect to powerful ray patterns. Current research efforts are focused on understanding how the two approaches discussed above might be modified to make them fruitful for the reducible case.

# Bibliography


[1] L. Bassett, J. Maybee, and J. Quark, *The correspondence principle in a qualitative environment*, Econometrica, 36 (1968), pp. 544-563.

[2] R.A. Brualdi and H.J. Ryser, *Combinatorial Matrix Theory*, Cambridge University Press, New York, 1991.

[3] C. Eschenbach, F. Hall, C.R. Johnson, and Z. Li, *The graphs of unambiguous entries in the product of two* $(+,-)$ *sign patterns*, Linear Algebra Appl., 260 (1997), pp. 95-118.

[4] G. M. Lady and J. Maybee, *Qualitatively invertible matrices*, Math. Social Sci., 6 (1983), pp. 397-407.

[5] Z. Li, F. Hall, and C. Eschenbach, *On the period and base of a sign pattern matrix*, Linear Algebra Appl., 212/2134 (1994), pp. 101-120.

[6] Z. Li, F. Hall, and J. Stuart, *Irreducible powerful ray pattern matrices*, Linear Algebra Appl., 342 (2002), pp. 47-58.

[7] J.J. McDonald, D.D. Olesky, M. Tsatsomeros, and P. van den Driessche, *Ray patterns of matrices and nonsingularity*, Linear Algebra Appl., 267 (1997), pp. 359-373.

[8] J. Stuart, *Determinants of hessenberg* $L$*-matrices*, SIAM J. Matrix Anal. Appl., vol. 12, no. 1 (1991), pp. 7-15.

[9] J. Stuart, *Reducible, sign* $k$*-potent sign pattern matrices*, Linear Algebra Appl., 294 (1999), pp. 197-211.

[10] J. Stuart, *Reducible pattern* $k$*-potent ray pattern matrices*, Linear Algebra, Appl., to appear.

[11] J. Stuart, L. Beasley, and B. Shader, *Irreducible, pattern* $k$*-potent ray pattern matrices*, Linear Algebra Appl., 346 (2002), pp. 261-271.

[12] J. Stuart, C. Eschenbach, and S. Kirkland, *Irreducible, sign* $k$*-potent sign pattern matrices*, Linear Algebra Appl., 294 (1999), pp. 85-92.

[13] J. Stuart and J. Weaver, *Matrices that commute with a permutation*, Linear Algebra Appl., 150 (1991), pp. 255-265.