Fourier Model Reduction for Large-Scale Applications in Computational Fluid Dynamics

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A new method, Fourier model reduction (FMR), for obtaining stable, accurate, low-order models of very large linear systems is presented. The technique draws on traditional control and dynamical system concepts and utilizes them in a way which is computationally very efficient. Discrete-time Fourier coefficients of the large system are calculated and used to construct a reduced-order model that preserves stability properties of the original system. Many coefficients can be calculated, which results in a very accurate representation of the system dynamics, but only a single inversion of the large system is required. The resulting system can be further reduced using explicit formulae for balanced truncation. The method is applied to a computational fluid dynamic system that models unsteady flow in a supersonic diffuser and the results are excellent. In comparison with other widely used reduction techniques, the new method is computationally efficient, preserves the stability of the original system, uses both input and output information, and is valid over a wide range of frequencies.

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1 Introduction

Despite increasing computational resources, for many applications high-order, complicated numerical models are impractical. This is often the case when coupling between disciplinary models is required. For example, computational fluid dynamic (CFD) models are not appropriate for use in many aeroelastic applications or for active flow control design. The goal of model reduction is to systematically develop a low-order model that captures the relevant system dynamics accurately over a range of frequencies and forcing inputs. For very large systems, the cost of reduction is often a critical issue.

We consider the specific task of finding a low-order, stable, continuous time, linear time invariant (LTI), state-space model

$$\hat{G}: \quad \frac{d}{dt}\hat{x}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t), \qquad \hat{y}(t) = \hat{C}\hat{x}(t) + \hat{D}u(t) \tag{1}$$

which approximates well the given stable model

$$G: \quad E\frac{d}{dt}x = Ax + Bu, \qquad y = Cx + Du \tag{2}$$

In the above, $x \in \mathbf{R}^n$ and $\hat{x} \in \mathbf{R}^k$ represent the full order and reduced state vectors respectively, u contains the system inputs, y contains the outputs of the full order system, and $\hat{y}$ contains the outputs of the approximate model.

The quality of $\hat{G}$ as an approximation of G is defined as the H-Infinity norm of the difference between their transfer functions:

$$\|\hat{G} - G\|_\infty = \sup_{\omega \in \mathbf{R}} |\hat{G}(j\omega) - G(j\omega)| \tag{3}$$

No efficient (polynomial time) solution is known for the problem of minimizing $\|\hat{G} - G\|_\infty$ subject to the order and stability constraints imposed on $\hat{G}$. Algorithms such as optimal Hankel model reduction [1, 2, 7] and balanced truncation [9] have been widely used throughout the controls community to generate suboptimal reduced models with strong guarantees of quality. These algorithms can be performed in polynomial time; however, the computational requirements make them impractical for application to large systems such as those encountered in CFD applications where system orders typically exceed 10^4. Approximate balancing methods have been developed using Krylov subspace approaches [4, 6, 12], but have not been widely used throughout the fluids community.

Rather, the proper orthogonal decomposition (POD), also known as Karhunen-Loéve expansions, has developed as a popular method of deriving reduced-order models for CFD applications using the method of snapshots [11, 5]. Moment matching techniques such as the Arnoldi method, developed in the context of integrated circuit modeling [10], have also been used for CFD turbomachinery applications [13]. The POD and Arnoldi methods have been successfully applied for a broad range of fluid dynamic applications (see, for example, [3] for a review of POD applications in CFD); however, these methods lack many of the desirable properties possessed by methods such as optimal Hankel model reduction. In particular, neither the POD nor the Arnoldi method offers any guarantees on accuracy or stability of the reduced model. Although these techniques often work well for CFD applications, in practice, they sometimes yield unstable models despite the original system being stable. Moreover, neither method takes account of system outputs when performing the reduction, hence the reduced-order models produced may be inefficient.

In this paper, a very low complexity algorithm is introduced, which allows one to find an intermediate approximation of G by a model $\bar{G}$ of order in the range of hundreds. While $\bar{G}$ is not an optimal reduced model of G, it satisfies an attractive guaranteed H-Infinity quality bound. After $\bar{G}$ is found, a second round of more demanding model reduction, such as balanced truncation, is applied to $\bar{G}$ to produce a high-quality, low-order, reduced model $\hat{G}$.

The organization of the paper is as follows. In the following section, the dynamical system arising from implementation of CFD is briefly described. The Fourier model reduction (FMR) technique is then presented and applied to an example which considers the flow dynamics of a supersonic diffuser. Finally, conclusions are drawn.

2 Computational Fluid Dynamic Model

A general linearized CFD model can be written in the form (2), where $x(t)$ contains the n unknown perturbation flow quantities at each point in the computational grid. For example, for two-dimensional, compressible, inviscid flow, which is governed by the Euler equations, the unknowns at each grid point are the perturbations in flow density, Cartesian momentum components and flow energy. The definition of inputs, $u(t)$, and outputs, $y(t)$, will depend upon the problem at hand. In aeroelastic analysis of a wing, inputs consist of wing motion while outputs of interest are the forces and moments generated. For control purposes, the output might monitor a flow condition at a particular location which varies in response to a disturbance in the incoming flow.

In the case of a CFD model, the linearization matrices A, B, C, D and E in (2) are evaluated at steady-state flow conditions. Typically, A and E are sparse, square matrices of very large dimension $n > 10^4$. The descriptor matrix E is included for generality, and may contain some zero rows, which arise from implementation of flow boundary conditions. On solid walls, a condition is imposed on the flow velocity, while at farfield boundaries certain flow parameters are specified, depending on the nature of the boundary (inflow/outflow) and the local flow conditions (subsonic/supersonic). Although these prescribed quantities could be condensed out of (2) to obtain a smaller state-space system, such a manipulation is often complicated and can destroy the sparsity of the system. The more general form of the system is therefore considered.

Typically, for applications such as aeroelasticity and active flow control, the desired order k of $\hat{G}$ is less than 50. In the next section, we present an efficient method with quality guarantees to reduce the size of the system while retaining an accurate representation of important flow dynamics.

3 Fourier Series Model Reduction

3.1 Fourier Series of Discrete Time Systems

Consider the full DT LTI system model g defined by the difference equations

$$g: \quad x(t+1) = ax(t) + bu(t), \quad y(t) = cx(t) + du(t), \tag{4}$$

where a, b, c, d are given matrices of coefficients, $x(t) \in \mathbf{R}^n$ is the system state, and $u(t), y(t)$ are input and output vectors. It will be assumed that g is stable, i.e. $\rho(a) < 1$, where $\rho(M)$ denotes the *spectral radius* of M, defined as the maximal absolute value of its eigenvalues.

The transfer function

$$g(z) = d + c(zI - a)^{-1}b \tag{5}$$

has the Fourier decomposition

$$g(z) = \sum_{k=0}^{\infty} g_k z^{-k}, \tag{6}$$

where

$$g_0 = d, \qquad g_k = ca^{k-1}b \ (k = 1, 2, \ldots) \tag{7}$$

The Fourier expansion converges exponentially for $|z| > \rho(a)$. Note that the first m Fourier coefficients g_k are easy to calculate using the "cheap" iterative process

$$g_k = ch_{k-1}, \ h_k = ah_{k-1} \ (k = 1, \ldots, m), \quad \text{where } h_0 = b, \tag{8}$$

which is expected to be "stable" since $\rho(a) < 1$.

Let $\hat{g}_m$ denote the m^{th} order approximation of g based on the Fourier series expansion:

$$\hat{g}_m(z) = \sum_{k=0}^{m} g_k z^{-k}. \tag{9}$$

The following simple result provides an estimate of the approximation error in the scalar case

$$\|g - \hat{g}_m\|_\infty = \max_{|z|=1} |g(z) - \hat{g}_m(z)|, \tag{10}$$

which ties it to the smoothness of G as follows.

Theorem 1. *For* $q = 1, 2, \ldots$

$$\|g - \hat{g}_m\|_\infty^2 \leq \frac{m^{1-2q}}{2\pi(2q-1)} \int_{-\pi}^{\pi} |g^{(q)}(e^{j\tau})|^2 d\tau,$$

where $g^{(q)}$ *is the* q^{th} *derivative of* g *with respect to* τ.

Proof:

$$|g(z) - \hat{g}_m(z)|^2 = \left| \sum_{k=m+1}^{\infty} g_k z^{-k} \right|^2$$

$$= \left| \sum_{k=m+1}^{\infty} g_k k^q z^{-k} k^{-q} \right|^2$$

Using the Cauchy inequality and considering $z = e^{j\tau}$*,*

$$\begin{aligned}
\left|g(e^{j\tau}) - \hat{g}_m(e^{j\tau})\right|^2 &\leq \left(\sum_{k=m+1}^{\infty} |g_k|^2 k^{2q} \right) \left(\sum_{k=m+1}^{\infty} k^{-2q} \right) \\
&\leq \left(\sum_{k=0}^{\infty} |g_k|^2 k^{2q} \right) \left(\int_m^{\infty} \frac{dx}{x^{2q}} \right)
\end{aligned}$$

Using the fact that

$$\left(\frac{\partial}{\partial \tau} \right)^q g\left(e^{j\tau}\right) = \sum_{k=0}^{\infty} (-jk)^q g_k e^{-jk\tau}$$

and applying Parseval's theorem, we obtain

$$|g(e^{j\tau}) - \hat{g}_m(e^{j\tau})|^2 \leq \frac{m^{1-2q}}{2\pi(2q-1)} \int_{-\pi}^{\pi} |g^{(q)}(e^{j\tau})|^2 d\tau$$

3.2 Fourier Series of Continuous Time Systems

Consider the full continuous time LTI system model G defined by the system (2). It will be assumed that G is stable, i.e. that all roots of the characteristic equation $\det(sE - A) = 0$ have negative real part, and that $C(sE - A)^{-1}B$ remains bounded as $s \to \infty$.

Let $\omega_0 > 0$ be a fixed positive real number. The transfer function

$$G(s) = D + C(sE - A)^{-1}B \tag{11}$$

has the Fourier decomposition

$$G(s) = \sum_{k=0}^{\infty} G_k \left(\frac{s - \omega_0}{s + \omega_0} \right)^k, \tag{12}$$

where

$$G_0 = d, \quad G_k = ca^{k-1}b \; (k = 1, 2, \ldots), \tag{13}$$

$$d = D + C(\omega_0 E - A)^{-1}B, \tag{14}$$

$$a = -(\omega_0 E + A)(\omega_0 E - A)^{-1}, \tag{15}$$

$$c = 2\omega_0 C(\omega_0 E - A)^{-1}, \tag{16}$$

$$b = -E(\omega_0 E - A)^{-1}B. \tag{17}$$

The Fourier expansion is derived from the identity

$$G(s) = g(z) = d + c(zI - a)^{-1}b \;\text{ for }\; z = \frac{s + \omega_0}{s - \omega_0},$$

which allows one to apply the observations and theorem from the previous subsection to this case. Note that by comparing (6) and (12), it can be seen that $g_k = G_k$.

3.3 Model Reduction Algorithm

The FMR algorithm is summarized in the following steps. Note that an effective approach is to use the efficient iterative procedure to derive an intermediate reduced model with several hundred states (at the cost of one n^{th}-order matrix inversion or factorization) and then perform a further reduction using balanced truncation.

1. Choose a value of ω_0. The value of ω_0 should reflect the frequency range of interest. The nominal value is unity; however, if the response at high frequencies is of interest, a higher value of ω_0 should be chosen. One can visualize the transformation from continuous to discrete time as a mapping of the imaginary axis in the $s-$plane to the unit circle in the $z-$plane. The value of ω_0 then describes the compression of frequencies around the unit circle.

2. Calculate $m+1$ Fourier coefficients using (13)-(17). Using the iterative procedure, any number of coefficients can be calculated with a single n^{th}-order matrix inversion or factorization.

3. Use the Fourier coefficients, $g_0, g_1, ..., g_m$, to construct an m^{th}-order DT reduced model

$$\hat{g}: \quad \hat{x}[t+1] = \hat{a}\hat{x}[t] + \hat{b}u[t], \qquad \hat{y}[t] = \hat{c}\hat{x}[t] + \hat{d}u[t], \tag{18}$$

   where

$$\hat{a} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \ddots & \end{bmatrix} \quad \hat{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad \hat{c} = [\, g_1 \quad g_2 \quad \ldots \quad g_m \,] \quad \hat{d} = g_0 \tag{19}$$

4. If further reduction is desired, balanced truncation can be applied efficiently to the DT system, since the expressions for the grammians are known explicitly. For the DT reduced model constructed in step 3, the controllability matrix is the identity matrix and the observability matrix is the Hankel matrix that has $\hat{c}$ as its first row. The balancing vectors can therefore be obtained by computing the singular vectors of the m^{th}-order Hankel matrix

$$\Gamma = \begin{bmatrix} g_1 & g_2 & g_3 & \cdots & g_{m-1} & g_m \\ g_2 & g_3 & g_4 & \cdots & g_m & 0 \\ g_3 & g_4 & g_5 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ g_{m-1} & g_m & 0 & \ldots & 0 & 0 \\ g_m & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}. \tag{20}$$

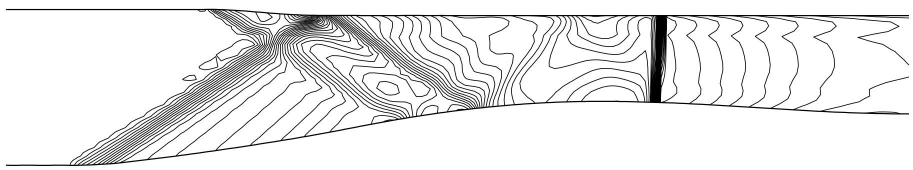

Figure 1. *Mach contours for steady flow through supersonic diffuser. Steady-state inflow Mach number is 2.2.*

The Hankel singular values, $\sigma_i, \; i = 1, 2, ..., m$, of the intermediate reduced system are given by the singular values of Γ.

5. Convert the low-order, DT reduced model to a continuous-time model using the relationships

$$\hat{A} = \omega_0 \left(\hat{a} - I\right)^{-1} \left(\hat{a} + I\right), \tag{21}$$

$$\hat{B} = 2\omega_0 \left(\hat{a} - I\right)^{-1} \hat{b}, \tag{22}$$

$$\hat{C} = -\hat{c} \left(\hat{a} - I\right)^{-1}, \tag{23}$$

$$\hat{D} = \hat{d} - \hat{c} \left(\hat{a} - I\right)^{-1} \hat{b} \tag{24}$$

We note that in the general case of a system with p inputs and q outputs, the algorithm above still applies. In this case, a total of pqm Fourier coefficients must be calculated, however this computation is very efficient using the iterative process in (8), since the matrix a does not depend on the inputs or outputs. Once the Fourier coefficients have been calculated, one can construct directly the Hankel matrix given by (20), where each component g_k is now a matrix of size $q \times p$.

4 Results

The test case considered in this paper is unsteady flow through a supersonic diffuser. The CFD model is developed by linearizing the Euler equations about a steady-state solution, the Mach contours for which are plotted in Figure 1. The nominal inflow Mach number for this steady case is 2.2. The CFD formulation for this problem results in a system of the form (2) and is described fully in Lassaux [8]. The CFD model has 3078 grid points and 11,730 unknowns.

A reduced-order model is required for the diffuser dynamics in order to derive active control strategies. These strategies will be used to counteract the effect of variations in the incoming flow. In nominal operation, there is a strong shock downstream of the diffuser throat, as can be seen in Figure 1. Incoming disturbances can cause the shock to move forward towards the throat. When the shock sits at the throat, the inlet is unstable, since any disturbance that moves the shock slightly upstream will cause it to move forward rapidly, leading to unstart of the inlet. This is extremely undesirable, since unstart results in a large loss of thrust. In order to prevent unstart from occurring, flow bleeding upstream of the diffuser throat will be used to actively control the position of the shock.

Figure 2 shows the transfer function between bleed actuation and average throat Mach number. The CFD results are compared to FMR with $m = 200$ and $\omega_0 = 1, 5, 10$. The frequency axis is non-dimensionalized by $f_0 = a_0/h$, where a_0 is the freestream speed of sound and h is the height of the diffuser. The reduced-order models can be seen to capture the dynamics very accurately over a wide range of frequencies. Typical disturbances are on the range $[0, 2f_0]$, however this model also captures higher frequencies should they occur. At higher frequencies, a slight error is observed for the $\omega_0 = 1$ model, however $\omega_0 = 5, 10$ yield very accurate results. This discrepancy highlights the importance of selecting a frequency scaling factor appropriately. Note that a frequency $\omega_0 = 1$ corresponds to $f/f_0 = 1/2\pi$.

While $m = 200$ represents a significant reduction in order from the CFD, a model of this size is too large for derivation of active control strategies. However, the explicit formulae for balanced truncation can now be applied to further reduce the size of the model. The first thirty Hankel singular values of the intermediate reduced model are plotted in Figure 3 for the $\omega_0 = 5$ case. Figure 3 indicates that at least ten states should be retained to achieve an accurate representation. The transfer functions of reduced-order models with five and ten states are compared against CFD in Figure 4. While the model with five states has some error, with just ten states the results are almost indistinguishable from the CFD.

FMR is also applied to the transfer function between an incoming density perturbation and the average throat Mach number. This transfer function represents the dynamics of the disturbance to be controlled and is shown in Figure 5. As the figure shows, the dynamics contain a delay, and are thus more difficult for the reduced-order model to approximate. Results are shown for FMR with $m = 200$ and $\omega_0 = 5, 10$. With $\omega_0 = 5$, the model has significant error for frequencies above $f/f_0 = 2$. Choosing a higher value of ω_0 improves the fit, although some discrepancy remains. These higher frequencies are unlikely to occur in typical atmospheric disturbances, however if they are thought to be important, the model could be further improved by either evaluating more Fourier coefficients, or by choosing a higher value of ω_0. The $\omega_0 = 10$ model is further reduced via balanced truncation to a system with thirty states without a noticeable loss in accuracy.

5 Conclusions

Fourier model reduction (FMR) is a new method for model reduction of very large linear systems. The method yields accurate, guaranteed stable reduced models, which can be derived using an efficient iterative procedure. An effective use of the method is to derive an intermediate discrete-time reduced system, with more states than desired but with very accurate dynamics capture, and then to apply a second model reduction technique. Balanced truncation can be applied very efficiently to further reduce the system, since the system grammians are known explicitly. Efficient calculation of the intermediate system is a critical step in enabling the application of control-based theory to very large systems.

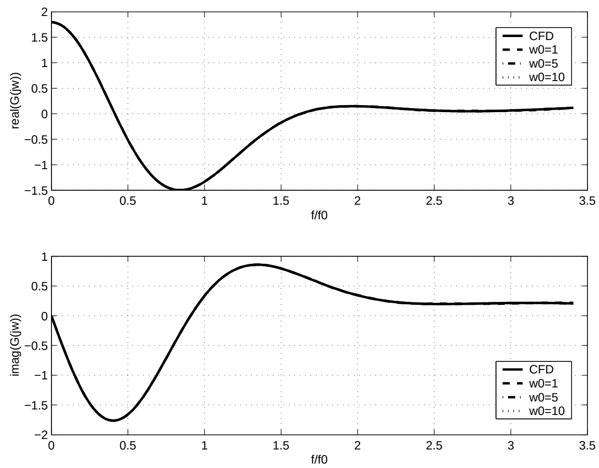

Figure 2. *Transfer function from bleed actuation to average throat Mach number for supersonic diffuser. Results from CFD model ($n = 11,730$) are compared to reduced-order models derived using 201 Fourier coefficients with $\omega_0 = 1, 5, 10$.*

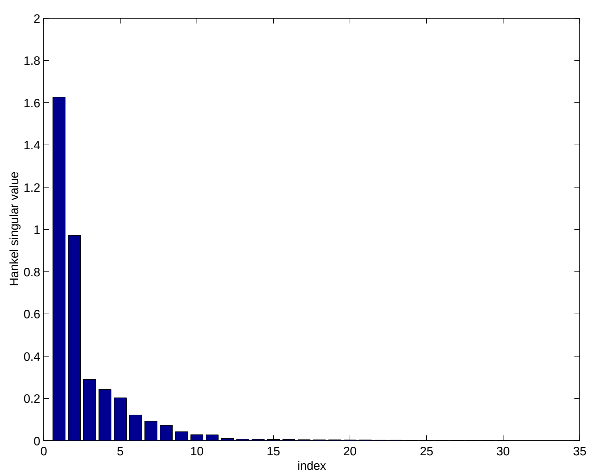

Figure 3. *Hankel singular values for supersonic diffuser case. Values are calculated using a 200^{th}-order model derived using FMR with $\omega_0 = 5$.*

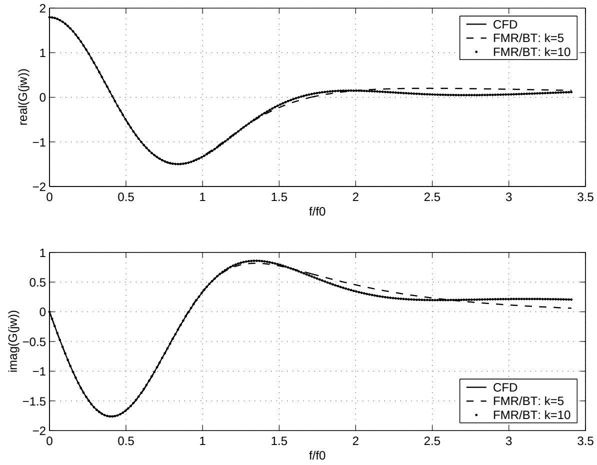

Figure 4. *Transfer function from bleed actuation to average throat Mach number for supersonic diffuser. Results from CFD model ($n = 11,730$) are compared to reduced-order models with five and ten states derived from the $m = 200$ FMR model via balanced truncation.*

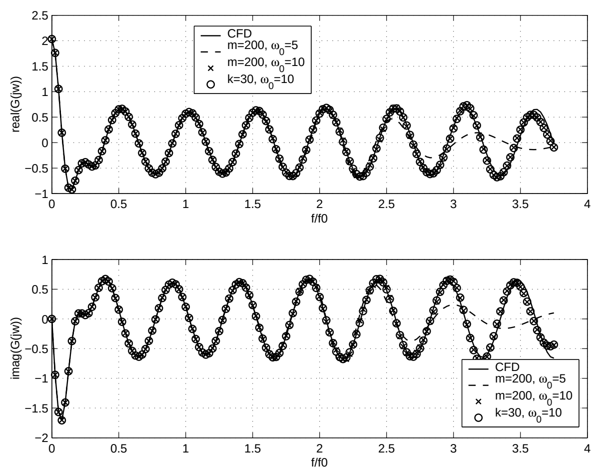

Figure 5. *Transfer function from incoming density perturbation to average throat Mach number for supersonic diffuser. Results from CFD model ($n = 11,730$) are compared to 200^{th}-order FMR models with $\omega_0 = 5, 10$. The $\omega_0 = 10$ model is further reduced to $k = 30$ via balanced truncation.*

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