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Cayley, Sylvester, and Early Matrix Theory

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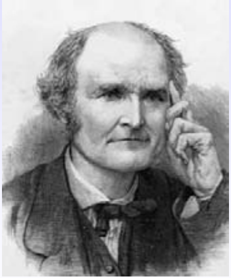
Cayley and Sylvester

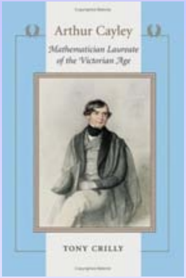
- Term **"matrix"** coined in 1850 by James Joseph Sylvester, FRS (1814–1897).

- **Matrix algebra** developed by Arthur Cayley, FRS (1821–1895).
  Memoir on the Theory of Matrices (1858).

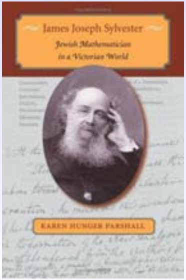


Biographies

Tony Crilly, *Arthur Cayley: Mathematician Laureate of the Victorian Age*, 2006.



Karen Hunger Parshall, *James Joseph Sylvester. Jewish Mathematician in a Victorian World*, 2006.



	Cayley	Sylvester
Enter Cambridge University	Trinity College, 1838	St. John’s College, 1831
Wrangler in Tripos examinations	Senior Wrangler, 1842	Second wrangler, 1837
Work in London	Pupil barrister from 1846; called to the Bar in 1849	Actuary from 1844; pupil barrister from 1846; called to the Bar in 1850
Elected Fellow of the Royal Society	1852	1839
President of the London Mathematical Society	1868–1869	1866–1867
Awarded Royal Society Copley Medal	1882	1880
Awarded LMS De Morgan Medal	1884	1887
British Assoc. for the Advancement of Science	President, 1883	Vice President, 1863–1865; President of Section A, 1869

Academic Positions

Cayley	Sylvester
• Sadleirian Chair, Cambridge 1863	• UCL, 1838 • U Virginia, 1841
	• Royal Military Academy, Woolwich, London, 1855
	• Johns Hopkins University 1876 • Savilian Chair of Geometry, Oxford, 1883

- Crilly (2006):

Statistically, Cayley's attention to matrix algebra is even slighter than his attention to group theory and is insignificant when compared to the large corpus he produced on invariant theory.

- Sylvester's work was mainly algebraic.
- Close friends: met around 1847.
- Cayley: widely read, well aware of other research in Britain and continent.
- Sylvester: mercurial, temperamental. Became involved in a number of disputes:

"Explanation of the Coincidence of a Theorem Given by Mr Sylvester in the December Nu mber of This Journal, With One Stated by Professor Donkin in the June Number of the Same"

Cayley's Notation (1855)

No. 3.

Remarques sur la notation des fonctions algébriques.

Je me sers de la notation

$$\begin{vmatrix} \alpha, & \beta, & \gamma, & \ldots \\ \alpha', & \beta', & \gamma', & \ldots \\ \alpha'', & \beta'', & \gamma'', & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

pour représenter ce que j'appelle une *matrice;* savoir un *système* de quantités rangées en forme de *carré*, mais d'ailleurs tout à fait *indépendantes* (je ne parle pas ici des *matrices rectangulaires*). Cette notation me parait très commode pour la théorie des équations *linéaires;* j'écris par ex:

$$(\xi, \eta, \zeta \ldots) = \begin{vmatrix} \alpha, & \beta, & \gamma & \ldots \\ \alpha', & \beta', & \gamma' & \ldots \\ \alpha'', & \beta'', & \gamma'' & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} (x, y, z \ldots)$$

Cayley's 1858 Memoir (1)

Introduces addition, multiplication, inversion, powering of matrices; zero and identity matrices.

52. A matrix such as

$$\left(\begin{array}{ccc} a, & b, & c \\ a', & b', & c' \end{array}\right)$$

where the number of columns exceeds the number of lines, is said to be a broad matrix; a matrix such as

$$\left(\begin{array}{cc} a, & b \\ a', & b' \\ a'', & b'' \end{array}\right)$$

where the number of lines exceeds the number of columns, is said to be a deep matrix.

Cayley's 1858 Memoir (2)

Cayley–Hamilton theorem: $p(t) = \det(tI - A)$ implies $p(A) = 0$. Proved for $n = 2$:

> *I have not thought it necessary to undertake the labour of a formal proof of the theorem in the general case of a matrix of any degree.*

More General Cayley–Hamilton Theorem

Theorem (Cayley, 1857)

If $A, B \in \mathbb{C}^{n\times n}$, $AB = BA$, *and* $f(x, y) = \det(xA - yB)$ *then* $f(B, A) = 0$.

Cayleys Memoir: Matrix Functions

- Finds some square roots of 2×2 matrices.
  And later in 1872:

$$X = \frac{A + \sqrt{\det(A)}\, I}{\sqrt{\mathrm{trace}(A) + 2\sqrt{\det(A)}}}.$$

- Finds parametrized family of 3×3 involutory matrices.

> *It is nevertheless possible to form the powers (positive or negative, integral or fractional) of a matrix, and thence to arrive at the notion of a rational and integral function, or generally of any algebraical function, of a matrix.*

Sylvester's Words

- **annihilator**, **canonical form**, **discriminant**, **Hessian**, **Jacobian**, **minor**, **nullity**.
- **Latent root**:

  > It will be convenient to introduce here a notion
  > (which plays a conspicuous part in my
  > new theory of multiple algebra),
  > namely that of the *latent roots* of a matrix
  > —latent in a somewhat similar sense as vapour may be
  > said to be latent in water or smoke in a tobacco-leaf.

- **Derogatory**, also called **privileged**!

More Cayley Notation

Cayley introduced

$$AB^{-1} \equiv \begin{array}{|c} \multicolumn{1}{c|}{A} \\ \hline B \end{array}, \qquad B^{-1}A \equiv \begin{array}{c|} \multicolumn{1}{|c}{A} \\ \hline B \end{array}$$

and, in an 1860 letter to Sylvester,

$$AB^{-1} \equiv \begin{array}{c} A \\ \sim \\ B \end{array}, \qquad B^{-1}A \equiv \begin{array}{c} A \\ \backsim \\ B \end{array}.$$

Taber (1890) later suggested

$$A \colon B, \qquad \frac{A}{B}.$$

Hensel (1928) suggested A/B and $B\backslash A$.

Influence of the Memoir

Crilly (2006): that

> *Cayley's 'Memoir,' which could have been a useful starting point for further developments, went largely ignored . . . His habit of instant publication and not waiting for maturation had the effect of making the idea available even if it was effectively shelved.*

Sylvester on Matrix Theory

Sylvester worked on theory of matrices 1882–1884.

While teaching theory of substitutions, Sylvester

> *"lectured about three times, following the text closely and stopping sharp at the end of the hour. Then he began to think about matrices again. 'I must give one lecture a week on those,' he said. He could not confine himself to the hour, nor to the one lecture a week. Two weeks were passed, and Netto was forgotten entirely and never mentioned again."*

Sylvester's Contributions

- Investigated the matrix equation

$$AX^2 + BX + C = 0, \qquad A, B, C \in \mathbb{C}^{n\times n}.$$

in several papers published in the 1880s.

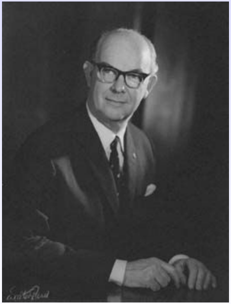
- Sylvester's law of inertia.
- Sylvester equation $AX + XB = C$.
- Gave first general definition of $f(A)$.

Recommended Reference

K. H. Parshall.
Joseph H. M. Wedderburn and the structure theory of algebras.
***Archive for History of Exact Sciences*, 32(3-4):223–349, 1985.**

Matrices in Applied Mathematics

- Frazer, Duncan & Collar, Aerodynamics Division of NPL: aircraft flutter, matrix structural analysis.
- **Elementary Matrices & Some Applications to Dynamics and Differential Equations, 1938**. Emphasizes importance of e^A.
- Arthur Roderick Collar, FRS (1908–1986): *"First book to treat matrices as a branch of applied mathematics".*



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