Interaction between internal and surface waves in a two-layers fluid

Catherine Sulem

University of Toronto

SIAM conference
Nonlinear Waves and Coherent Structures
August 16-19 2010

Joint work with W. Craig (McMaster) and P. Guyenne (U. of Delaware)

W. Craig, P. Guyenne, C. S., *Coupling between internal and surface waves*, Natural Hazards, Special Issue on ”Internal waves in the oceans and estuaries: modeling and observations” 2010.

X. Liu (U. Toronto)

Physical context

In the situation in which a fluid domain, such as the sea, consists of essentially two immiscible layers separated by a sharp interface such as a thermocline (sharp variation in temperature or a pycnocline (sharp variation of salinity), very large amplitude and long wavelength nonlinear waves can be produced in the interface and can propagate over large distances. For example, there are generated when tides cause water to move over submerged mountains on the ocean floor. Cold water from the bottom gets pushed up over the ridge and sets up a disturbance.

Among the most striking of early measurements were those of Perry-Schimke (1965) in the Andaman sea (eastern side of Indian Ocean). They found groups of internal waves up to 80m high, 2000 m long on the main thermocline at 500 m in water 1500 m deep.

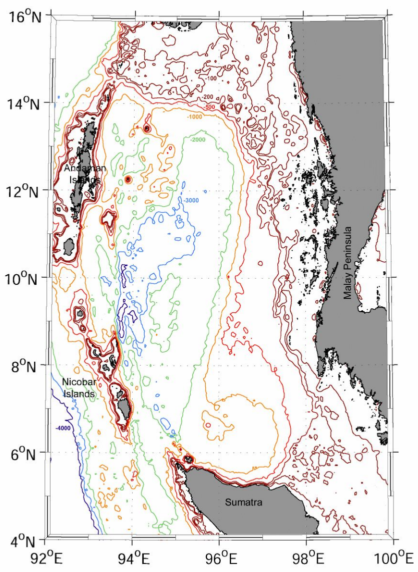


Figure: Bathymetry map of the Andaman sea

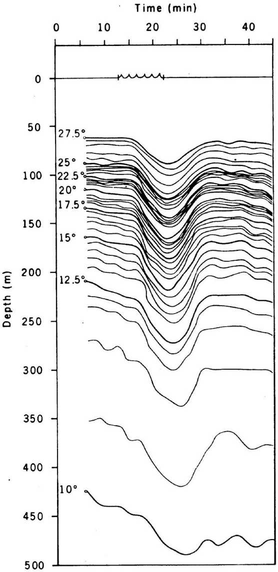


Figure: Isotherm contours on Oct. 25, 1976; Internal wave is materialized by isotherms. Osborne-Burch, 'Internal Solitons in the Andaman Sea', Science, **208** 1980, 451.

The combination of in situ and remote sensing observations, as well as the progress in detection technology over the last 40 years, have shown that internal soliton-like waves are important and common features of costal oceans in many regions of the world.
On the practical side, internal waves can significantly influence measurements of currents, undersea navigation, submerged engineering construction ... They play an important role in mixing different layers of water in the ocean which can affect the climate.
However, they are not directly visible to the observer, but they may produce in some instances small scale patterns at the surface that appears as a strip of rough waters.

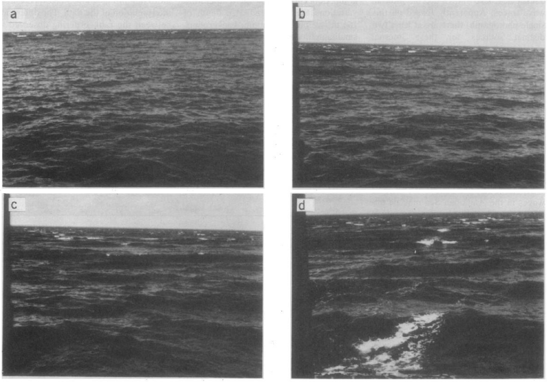


Figure: Sequence of photographs of the Andaman sea surface taken from an observation vessel (Oct. 27, 1976) as a rip band approches from the west at a speed of 2.2m/sec (Osborne-Burch 1980) ; (a) The rip is seen in the distance, stretching from one horizon to the other as a *well defined line of breaking waves*. The background sea state preceding the rip was $\sim$ 0.6 m. (b) continues to approach; (c) the rip has just arrived at the vessel with wave heights 1.8 m. (d) the vessel was tossed about in the 1.8m waves.

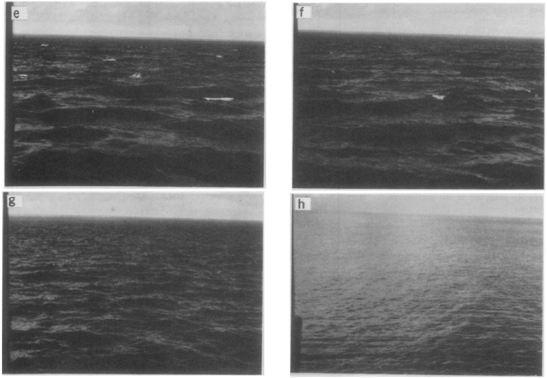


Figure: (e) the rearward edge of the rip was visible in 1.8 m waves; (f) the rearward edge of the rip receded as the wave dropped to 1.3 m; (g) the wave amplitudes have dropped to 0.6 m; (h) the rip has completely passed as waves dropped to ripples of 0.1 m.

Internal wave signatures have been observed in photographs taken from the space shuttle. The ripples induced by the internal waves have been imaged under the highly incident light of late afternoon. Their presence give rise to a differential reflectancy property under oblique lighting.

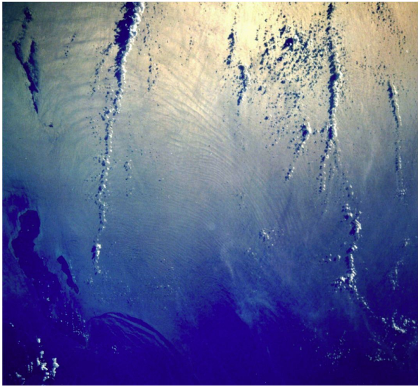


Figure: photograph taken on May 5, 1985; from *Atlas of Oceanic internal solitary waves, the Andaman sea; Office of Naval Research, 2002.*

On a historical note, there is a description of such a phenomenon in a book by F.M. Maury : *‘Physical Geography of the sea and its meteorology’* (1885) (quoted in Obsorne-Burch 1980)

In the entrance of the Malacca Straits, near the Nicobar and Acheen Islands, and between them and Junkseylon, there are often very strong ripplings, particularly in the southwest monsoon; these are alarming to persons unacquainted, for the broken water makes a great noise when the ship is passing through the ripplings in the night. In most places, ripplings are thought to be produced by strong currents, but here they are frequently seen when there is no perceptible current..... so as to produce an error in the course and distance sailed, yet the surface of the water is impelled foreward by some indiscovered cause.

The ripplings are seen in calm weather approaching from a distance, and in the night their noise is heard a considerable time before they come near. They beat against the sides of the ship with great violence, and pass on , the spray sometimes coming on deck; and a small boat could nit always resist the turbulence of these remarkable ripplings.

- Other fascinating pictures include those of the *Strait of Gibraltar* (where the Atlantic Ocean meets with the Mediterranean Sea). The two layers of fluid correspond to different salinity and the current is caused by the tides passing through the Strait.

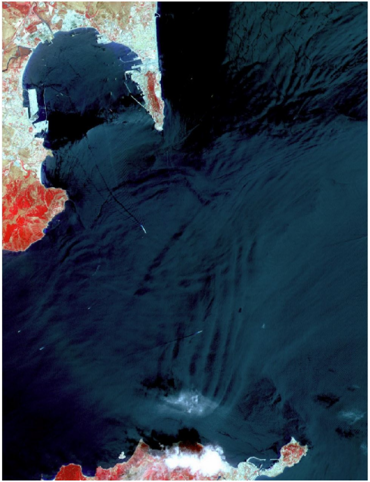
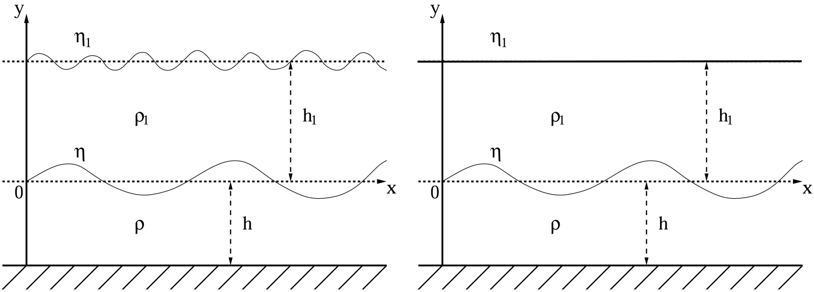


Figure: Strait of Gibraltar; from *the Atlas of Oceanic internal solitary waves, Office of Naval Research*

- An extensive collection of measurements and images of various regions in the world can be found at *http://www.internalwaveatlas.com*.
- Recent survey article by Helfrich and Melville (Ann. Rev.Fluid Mech 2006) with an overview of properties of internal solitary waves and vast bibliography.
- Finally internal waves are also generated in the atmosphere when winds blow over mountain ranges; *morning glory clouds (Australia)*.

Mathematical Models

Due to its importance in oceanography, there has been a large literature on internal waves in a variety of scaling regimes, and thus a variety of mathematical models.
2 physical settings : fixed lid, or internal/surface wave coupling,



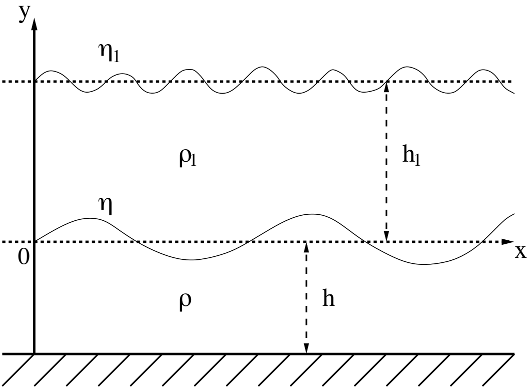
- Stable configuration : $\rho > \rho_1$; ρ_1/ρ close to 1
- layer thickness ratio h_1/h plays important role.

- Fix lid : Weakly nonlinear models for interface (Boussinesq, KdV, BO, ILW) ; fully nonlinear models Benjamin '67, Ono, '75, Camassa-Choi '96, '06, Nguyen-Dias '07, Bona-Lannes-Saut '07
- coupling interface/free surface: Long wave/long wave Gear-Grimshaw '84, Matsuno '93, Craig-Guyenne-Kalisch '05, Barros-Gravilyuk-Teshukov, '07;
- Fix lid, internal wave propagation over periodic bottom topography Ruis de Zárate-Vigo-Alfaro-Nachbin-Choi 2009

Long wave/short wave interaction

- I would like to focus on today on a regime that displays features of the pictures shown earlier: i.e. a free surface displaying small rough ripples created by the presence of a relatively large interface.
- Long wave regime for the interface, and ‘small’ quasi-monochromatic wave obeying the modulational Ansatz for the surface (Hashizume ’80).
- There is a clear scale separation.
- Goal: Write a mathematical model

The Euler Equations for stratified potential flow.



$$
\begin{aligned}
\Delta\varphi &= 0\,, \quad \text{in the lower domain } S(t;\eta) \\
\Delta\varphi_1 &= 0\,, \quad \text{in the upper domain } S_1(t;\eta,\eta_1)\,.
\end{aligned}
$$

Boundary conditions

On the fixed bottom $\{y = -h\}$ of the lower fluid, the boundary condition is

$$\partial_y \varphi(x, -h) = 0 \ ,$$

On the interface $\{y = \eta(x, t)\}$, three boundary conditions - 2 kinematic , 1 dynamic (Bernouilli):

$$\begin{aligned}
&\partial_t \eta = \partial_y \varphi - \partial_x \eta \, \partial_x \varphi \\
&\partial_t \eta = \partial_y \varphi_1 - \partial_x \eta \, \partial_x \varphi_1 \\
&\rho(\partial_t \varphi + \frac{1}{2}|\nabla \varphi|^2 + g\eta) = \rho_1(\partial_t \varphi_1 + \frac{1}{2}|\nabla \varphi_1|^2 + g\eta) \ ,
\end{aligned}$$

On the top free surface $\{y = h_1 + \eta_1(x, t)\}$, 2 boundary conditions:

$$\begin{aligned}
&\partial_t \eta_1 = \partial_y \varphi_1 - \partial_x \eta_1 \, \partial_x \varphi_1 \\
&\partial_t \varphi_1 + \frac{1}{2}|\nabla \varphi_1|^2 + g\eta_1 = 0 \ .
\end{aligned}$$

The goal is to describe simultaneously the evolution of the free surface and free interface.

Hamiltonian Formulation

It is possible to write the system in the form of a *Hamiltonian system* where the canonical variables are obtained in analogy with methods of *classical mechanics*.

- If the absence of internal wave (one fluid), the canonical variables are (η, ξ) where
  η is the free surface
  $\xi = \varphi(x, \eta(x))$ the trace of the velocity potential on the free surface (Zakharov 1968).
  The water wave problem takes the form

$$\partial_t \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \delta_\eta H \\ \delta_\xi H \end{pmatrix} ,$$

- Hamiltonian = Total energy.

$$
\begin{aligned}
H &= \text{kinetic energy} + \text{potential energy} \\
&= \int \int_{-h+\beta(x)}^{\eta(x)} \frac{1}{2} |\nabla \varphi(x,y)|^2 \, dydx + \int \frac{g}{2} \eta^2(x) \, dx \\
&= \frac{1}{2} \int \varphi \frac{\partial \varphi}{\partial n} d\sigma + \int \frac{g}{2} \eta^2(x) \, dx \\
&= \int \frac{1}{2} \xi(x) G(\eta) \xi(x) \, dx + \int \frac{g}{2} \eta^2(x) \, dx \; .
\end{aligned}
$$

Dirichlet – Neumann operator for the fluid domain

$$
\xi \to G(\eta)\xi = \sqrt{(1+\eta_x^2)} \, \frac{\partial \varphi}{\partial n}\Big|_{y=\eta} \;,
$$

Choice of canonical variables follow *principles of classical mechanics*: Given a curve $\eta(\cdot, t)$ in configuration space, the Lagrangian given by

$$L := L(\eta, \dot{\eta}) = \text{kinetic energy} - \text{potential energy}$$

Rewrite the kinetic energy entirely in terms of $(\eta, \dot{\eta})$: Use the kinematic equation on free surface

$$\dot{\eta} = \partial_y \varphi - \partial_x \eta \partial_x \varphi = \sqrt{(1 + \eta_x^2)} \frac{\partial \varphi}{\partial n} = G(\eta)\xi$$

$$L(\eta, \dot{\eta}) = \int \frac{1}{2} \dot{\eta} G^{-1}(\eta) \dot{\eta} \, dx - \int \frac{g}{2} \eta^2(x) \, dx \ .$$

The Legendre transform will identify the coordinate canonically conjugate to η. Indeed,

$$\delta_{\dot{\eta}} L = G^{-1}(\eta) \dot{\eta}$$

which dictates that $\xi(x) = \varphi(x, \eta(x))$ is the appropriate choice.

- For stratified fluids, similar construction of canonical variables $(\eta, \xi, \eta_1, \xi_1)$.
  η is the interface
  $h_1 + \eta_1$ is the free surface
  $\xi = \rho\Phi - \rho_1\Phi_1, \quad \xi_1 = \rho_1\Phi_2$.
  Φ, Φ_1, Φ_2 are defined as
  $\Phi = \varphi(x, \eta(x))$
  $\Phi_1 = \varphi_1(x, \eta(x)),\ \Phi_2 = \varphi_1(x, h_1 + \eta_1(x))$,
  (Benjamin-Bridges, 1997).
- Hamiltonian = Total energy.

$$\partial_t \begin{pmatrix} \eta \\ \xi \\ \eta_1 \\ \xi_1 \end{pmatrix} \equiv J \nabla H = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \delta_\eta H \\ \delta_\xi H \\ \delta_{\eta_1} H \\ \delta_{\xi_1} H \end{pmatrix} ,$$

Scaling regime

h : typical depth of lower fluid
a : order of amplitude of interface
λ : order of wavelength of interface

- Long wave/small amplitude regime for interface (KdV)

$$\frac{h}{\lambda} = \varepsilon, \ \ \frac{a}{h} = \varepsilon^2$$

h_1 : typical depth of upper fluid
a_1 : order of amplitude of free surface

- (very) small amplitude for surface: (modulational regime)

$$\frac{a_1}{h_1} = \varepsilon_1, \ \varepsilon_1 = \varepsilon^{2+\alpha}, \ \alpha \geq 0$$

Linear analysis near fluid at rest

Linearized equations:

$$\partial_t \eta = \delta_\xi H^{(2)},\ \partial_t \xi = -\delta_\eta H^{(2)}$$
$$\partial_t \eta_1 = \delta_{\xi_1} H^{(2)},\ \partial_t \xi_1 = -\delta_{\eta_1} H^{(2)}\ .$$

the quadratic part $H^{(2)}$ of the Hamiltonian is given by ($D = -i\partial_x$)

$$H^{(2)} = \frac{1}{2}\int (\xi, \xi_1)\mathcal{A}(D)\begin{pmatrix}\xi\\ \xi_1\end{pmatrix} + g(\rho-\rho_1)\eta^2 + g\rho_1\eta_1^2$$

where $\mathcal{A}(D)$ is a 2×2 matrix of Fourier multipliers. The dispersion relation is given by the quadratic equation for ω^2

$$\begin{aligned}\omega^4 \quad &- \quad g\rho k\frac{1+\tanh(kh)\coth(kh_1)}{\rho\coth(kh_1)+\rho_1\tanh(kh)}\omega^2\\ &+ \quad g^2(\rho-\rho_1)k^2\frac{\tanh(kh)}{\rho\coth(kh_1)+\rho_1\tanh(kh)} = 0\ .\end{aligned}$$

2 solutions associated to 2 different modes of wave motion:
The branch $\omega^+(k)$ associated to the free surface
and the branch $\omega^-(k)$ associated to the interface.
For example,

$$\lim_{k\to\infty} \omega^+(k) = gk$$

(agrees with dynamics of free surface with no interface present)

$$\lim_{k\to\infty} \omega^-(k) = \frac{g(\rho-\rho_1)}{\rho+\rho_1}k$$

(agrees with asymptotics of dispersion relation for rigid lid)
When kh and $kh_1 \to 0$ (with ratio h/h_1 finite): phase speeds asymptotic to

$$(c^\pm)^2 = \frac{g}{2}\left(h + h_1 \pm \sqrt{(h-h_1)^2 + 4\frac{\rho_1}{\rho}hh_1}\right) .$$

Note that the phase velocity $(c^-)^2$ associated with the free interface (the 'slower' dispersion curve) is positive for $\rho > \rho_1$ (stable stratification). Also, for $\rho > \rho_1$, the 'faster' free surface phase velocity c^+ is slower than if there were no interface present.

Scalings, Ansatz and resonant condition

Roughly speaking...

- Long wave/small amplitude regime for interface (KdV)

$$\eta \sim \varepsilon^2 r(X,\tau);\ X = \varepsilon x,\ \tau = \varepsilon^3 t$$

- (very) small amplitude for surface: (modulational regime)

$$\eta_1 \sim \varepsilon_1 v(X,\tau_1) e^{i(k_0 x - \omega^+(k_0)t)} + \mathrm{c.c},\ \varepsilon_1 = \varepsilon^{2+\alpha},\ \tau_1 = \varepsilon^2 t$$

- Assume unidirectional motion for the interface at velocity c^-. The wavenumber k_0 is chosen such that wave packets on the free surface (moving at group velocity $\omega^+(k_0)$) move at same speed as interface:

$$\omega^{+\prime}(k_0) = c^-$$

'linear resonant condition' between internal and surface waves.

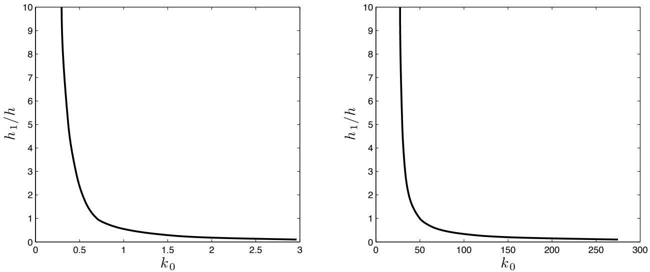


Figure: Depth ratio h_1/h vs. wavenumber k_0 corresponding to the linear resonance condition for $\rho_1/\rho = 0.1$ (left) and $\rho_1/\rho = 0.99$ (more realistic) (right) .

There is always a surface mode of wavenumber k_0 which satisfies the resonance condition and thus travels at the same linear speed as a long internal mode. The smaller h_1/h, or the closer ρ_1/ρ to unity, the larger k_0 (and hence the shorter the surface mode). In addition, k_0 varies monotonically as a function of h_1/h.

More precisely...... It is convenient to perform a normal mode decomposition

$$(\eta, \xi, \eta_1, \xi_1) \to (\mu, \zeta, \mu_1, \zeta_1)$$

so that the quadratic part of the Hamiltonian $H^{(2)}$ simplifies to

$$H^{(2)} = \frac{1}{2} \int \zeta \omega_-^2(D)\zeta + \mu^2 + \zeta_1 \omega_+^2(D)\zeta_1 + \mu_1^2 \, dx \ ,$$

Through this transformation, the equations of motion are transformed to

$$\partial_t \begin{pmatrix} \mu \\ \zeta \\ \mu_1 \\ \zeta_1 \end{pmatrix} = J \nabla H$$

Higher order terms of the Hamiltonian will be transformed as well.

Since both internal and surface wave propagate with their respective speeds, it is convenient to change the equations into a moving frame of reference. This is done by subtracting a multiple of the momentum I (Benjamin 1967)

$$
\begin{aligned}
I &= \int \Big(\rho \int_{-h}^{\eta(x)} \partial_x \varphi \, dy + \rho_1 \int_{\eta(x)}^{h_1+\eta_1(x)} \partial_x \varphi_1 \, dy\Big) dx \\
&= -\int \Big(\xi \partial_x \eta + \xi_1 \partial_x \eta_1\Big) dx = -\int \Big(\zeta \partial_x \mu + \zeta_1 \partial_x \mu_1\Big) dx \,,
\end{aligned}
$$

from the Hamiltonian, $H \to H - cI$. It is possible to do so because the total momentum is also a conserved quantity of the coupled system.

Long-wave scaling, modulational Ansatz and an additional canonical change of variables

We assume that the ‘internal’ modes are long waves according to the scalings

$$X = \varepsilon x \ , \quad \mu(x,t) = \varepsilon^2 \tilde{\mu}(X,t) \ , \quad \zeta(x,t) = \varepsilon \tilde{\zeta}(X,t) \ ,$$

the ‘surface’ modes are quasi-monochromatic waves obeying the modulational Ansatz, which after an additional canonical transformation takes the form ($\varepsilon_1 = \varepsilon^{2+\alpha}$)

$$\mu_1(x,t) = \frac{\varepsilon_1}{\sqrt{2}} \omega^+(D)^{1/2} \Big(v(X,t) e^{ik_0 x} + \mathrm{c.c} \Big) + \varepsilon_1^2 \tilde{\mu}_1(X,t) \ ,$$

$$\zeta_1(x,t) = \frac{\varepsilon_1}{\sqrt{2} i} \omega^+(D)^{-1/2} \Big(v(X,t) e^{ik_0 x} - \mathrm{c.c} \Big) + \frac{\varepsilon_1^2}{\varepsilon} \tilde{\zeta}_1(X,t) \ ,$$

The next step is to enter these scalings into the Hamiltonian and expand in powers of ε.....
Finally, we look at the dynamics of the system in a preferred direction of propagation by decomposing the interface into two components : $r(X, t)$ is the component that is principally right-moving, while $s(X, t)$ is principally left-moving.

Effective equations and interpretation

$(\tau = \varepsilon^3 t, \tau_1 = \varepsilon^2 t)$

$$\partial_\tau r + \lambda_1 r \partial_X r + \lambda_2 \partial_X^3 r = \varepsilon^{2\alpha} \lambda_3 \partial_X |v|^2$$
$$\partial_{\tau_1} v = i\Big[\frac{1}{2}\omega^{+\prime\prime}(k_0)\partial_X^2 v + \kappa r v\Big]$$

The coefficients $\lambda_1, \lambda_2, \lambda_3, k_0, \kappa$ depend on the parameters ρ, ρ_1, h, h_1. Choose $\alpha > 0$ so that rhs of KdV disappears.

We are interested in the situation in which the internal wave give rise to localized bound states for the linear Schrödinger equation : the surface wave patterns will exhibit trapped surface modes visible in the vicinity of the solution peak and will travel with it.

They are the effective signature of the presence of the internal waves.

Depending on the signs of the coefficients, the KdV equation $\partial_\tau r + \lambda_1 r \partial_X r + \lambda_2 \partial_X^3 r = 0$ has solitons which are depression or bumps.

Fix ρ_1/ρ close to 1 (for example 0.95), $h = 1$ and vary h_1.The coefficient λ_2 always > 0,

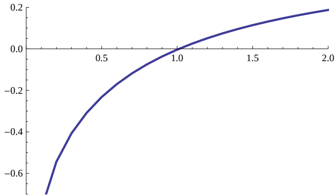


Figure: λ_1 versus h_1

$\lambda_1 < 0$ for h_1 small, $\lambda_1 > 0$ otherwise. If h_1 is sufficiently small, the KdV soliton $3u_0 \frac{\lambda_2}{\lambda_1} \operatorname{sech}^2(\frac{\sqrt{u_0}}{2}(X - u_0\lambda_2\tau))$ is a depression.

Turning to the Schrödinger equation with a (slowly varying) 1-soliton potential

$$\partial_{\tau_1} v = i\Big[\frac{1}{2}\omega^{+''}(k_0)\partial_X^2 v + \kappa r(X - \varepsilon u_0\tau_1)v\Big]$$

Make the change of variable $Y = X - \varepsilon u_0\tau_1$

$$\partial_{\tau_1} v - \varepsilon u_0 \partial_Y v = i\Big[\frac{1}{2}\omega^{+''}(k_0)\partial_X^2 v + \kappa r\ v\Big]$$

Phase shift $v = e^{i(pY+q\tau_1)}v_1$ to eliminate advection term; Look for solution $v_1 = e^{i\nu\tau_1}W$

$$-W'' - \frac{2\kappa}{\omega^{+''}(k_0)} rW = -\frac{2\nu}{\omega^{+''}(k_0)} W$$

Schrödinger operator: $-\partial_{xx} - \frac{2\kappa}{\omega^{+\prime\prime}(k_0)} r$. Fix ρ_1/ρ close to 1 (for example 0.95), $h = 1$ and vary h_1, we find $\kappa > 0$, $\omega^{+\prime\prime}(k_0) < 0$.

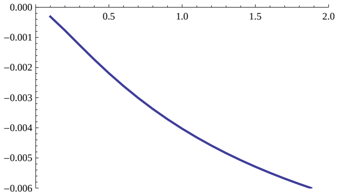
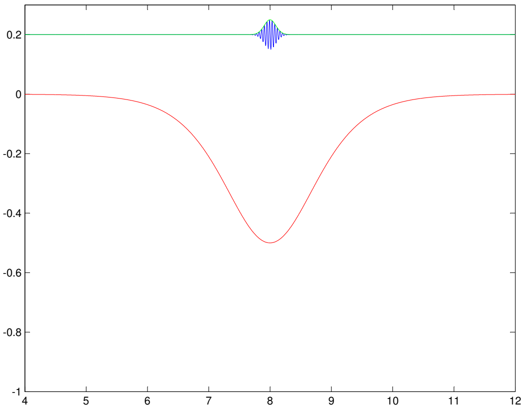


Figure: $\omega^{+\prime\prime}(k_0)$ versus h_1

For existence of bound states, we need $r < 0$, which corresponds to h_1 'small'. Note that $|\omega^{+\prime\prime}(k_0)|$ very small, like for the semi-classical limit. Bound states are very narrow concentrated close to the soliton.


0.2
0
-0.2
-0.4
-0.6
-0.8
-1
4
5
6
7
8
9
10
11
12

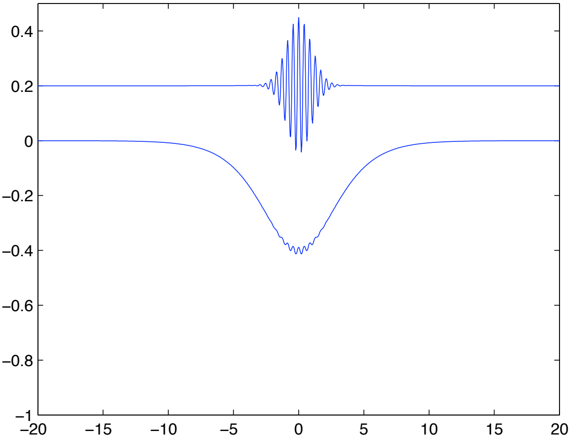


Figure: Physical variables

Conclusion

We have presented an asymptotic analysis of the coupling between the interface and the free surface of a two layer fluid, in a scaling regime in which the internal mode is treated as a long wavelength nonlinear internal wave, while the surface mode is smaller and taken in a modulational regime. This is a physically realistic situation for certain cases of internal waves in the ocean, whose visible signature on the surface is a band of roughness which propagates at the same velocity as the internal wave.

Using a perturbation theory we have derived a coupled set of equations which describe this regime, in which the internal mode evolves according to an equation of KdV type, and the surface mode is propagated at the resonant group velocity, and is modulated according to a time dependent linear Schrödinger equation.

In the case of a soliton internal wave (*when it is a wave of depression*), the Schrödinger equation will often have *bound states*, leading to the phenomenon of trapped surface wave modes which propagate as the signature of the internal wave. We propose this as a possible explanation for the bands of surface roughness observed in the pictures shown in the beginning, which are associated with the presence of large amplitude internal waves.

Another observation in the Osborne-Burch paper (1980) is that after the solitary wave and its ripples pass, the sea is very calm and flat 'like a mirror'. It is as though the internal wave sweeps up all the small scale surface disturbances into its bound states, which are then captive above the internal solitary wave and carried away with it. We do not have yet a mathematical model to describe this phenomenon.