Some Recent Advances in Mixed-Integer Nonlinear Programming

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An MINLP Research Initiative

- CMU-IBM research collaboration, started in 2004
- The Team:

CMU

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- Lorenz T. Biegler
- Gérard Cornuéjols
- Ignacio E. Grossmann
- Carl D. Laird (Texas A&M)
- François Margot
- Nick Sawaya
- Nick Sahinidis

IBM

- Pierre Bonami (CNRS Marseilles)
- Andrew R. Conn
- Claudia D'Ambrosio (U Bologna)
- John J. Forrest
- Joao Goncalves
- Oktay Günlük
- Laszlo Ladanyi
- Jon Lee
- Andrea Lodi (U Bologna)
- Andreas Wächter

Mixed-Integer Nonlinear Programming (MINLP)

$$\begin{aligned} \min \quad & f(x, y) \\ s.t. \quad & c(x, y) \le 0 \\ & y_L \le y \le y_U \\ & x \in \{0,1\}^n,\ y \in \mathbb{R}^p \end{aligned}$$

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- Often in practice: Simplify original problem to obtain
  - NLP by relaxing integrality conditions (rounding)
  - MILP by approximating nonlinearities (piece-wise linear)

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  - MILP by approximating nonlinearities (piece-wise linear)
- Goal: Design exact algorithms
- In this talk: Convex MINLP (f, c convex)

The Power Of MILP

- MILP has been extensively explored for decades
  - Based on branch-and-bound [Dakin (1965)]
  - Very powerful algorithms, techniques, and codes
  - Can solve very large problems
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Outer Approximation (Duran, Grossmann [1986])

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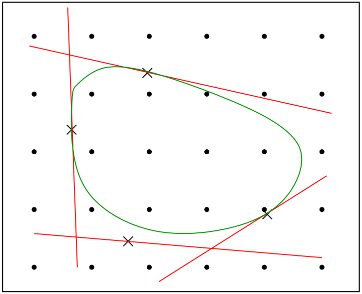

Approximate by MILP (hyperplanes)

$$
\begin{aligned}
\min \quad & z \\
s.t. \quad & \nabla f(x^k, y^k)^T \begin{pmatrix} x - x^k \\ y - y^k \end{pmatrix} + f(x^k, y^k) \leq z \\
& \nabla c(x^k, y^k)^T \begin{pmatrix} x - x^k \\ y - y^k \end{pmatrix} + c(x^k, y^k) \leq 0 \\
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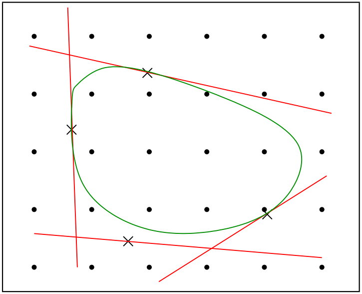

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- $\mathcal{T}$ contains linearization points
  - augmented during algorithm
- Algorithm: Repeat
  1. solve current MILP $\rightarrow (x^l, \tilde{y}^l)$
  2. solve NLP with x^l fixed $\rightarrow y^l$
  3. add (x^l, y^l) to $\mathcal{T}$

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- Original algorithm:
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  - Advantage: Simple to implement; uses all MILP techniques
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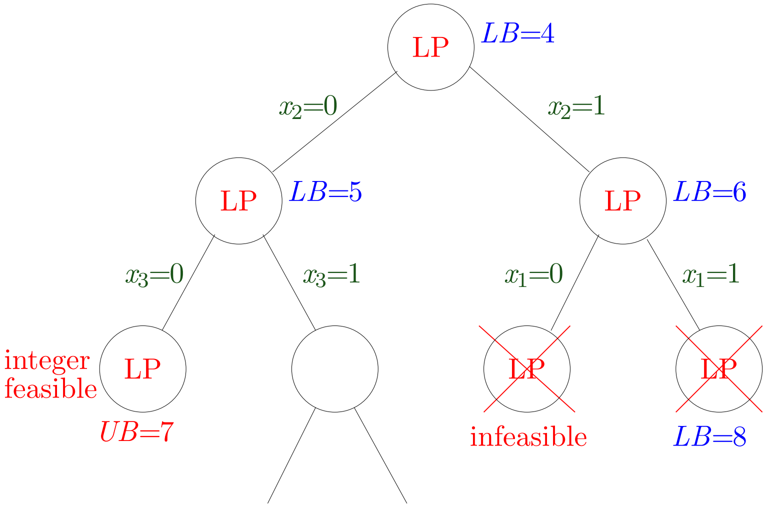

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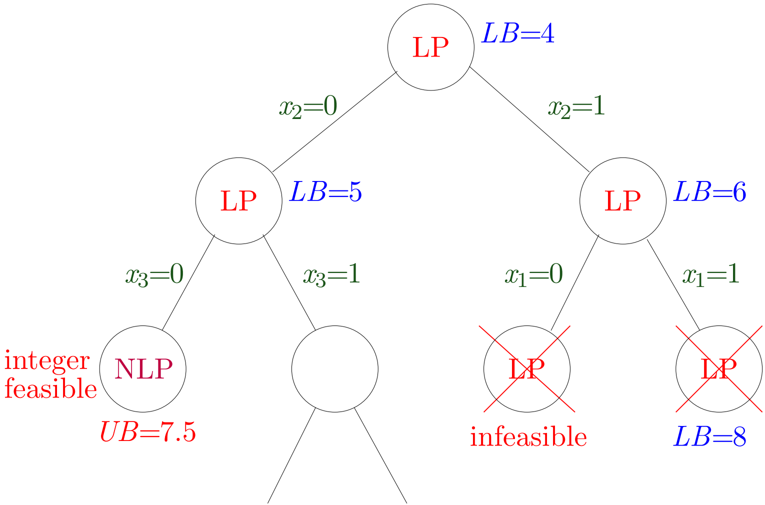

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- "Hybrid" approach [Bonami et al. (2005)]
  - Solve NLPs also at non-integer nodes
  - For example, solve NLP in every 10th node
    - \+ Includes information about nonlinear geometry more quickly
    - – Requires solution of more NLPs
  - Abhishek, Leyffer, Linderoth (2007) (`FilMINT` code):
    - ⋆ Don't solve NLP, just add linearization (Extended cutting plane)

Preliminary Numerical Experiments

- Software implementation
  - Bonmin (Open source software on COIN-OR)

    http://www.coin-or.org/Bonmin
  - Based on other COIN-OR projects (Cbc, Clp, Cgl, Ipopt, ...)

    - Essential for fast development: Availability of open source
  - NLP solvers: FilterSQP [Fletcher, Leyffer] and Ipopt

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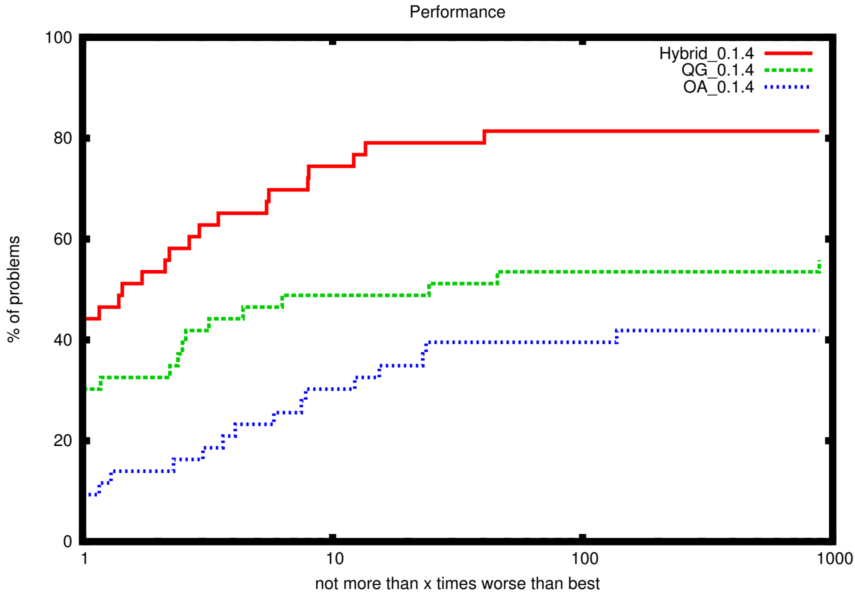
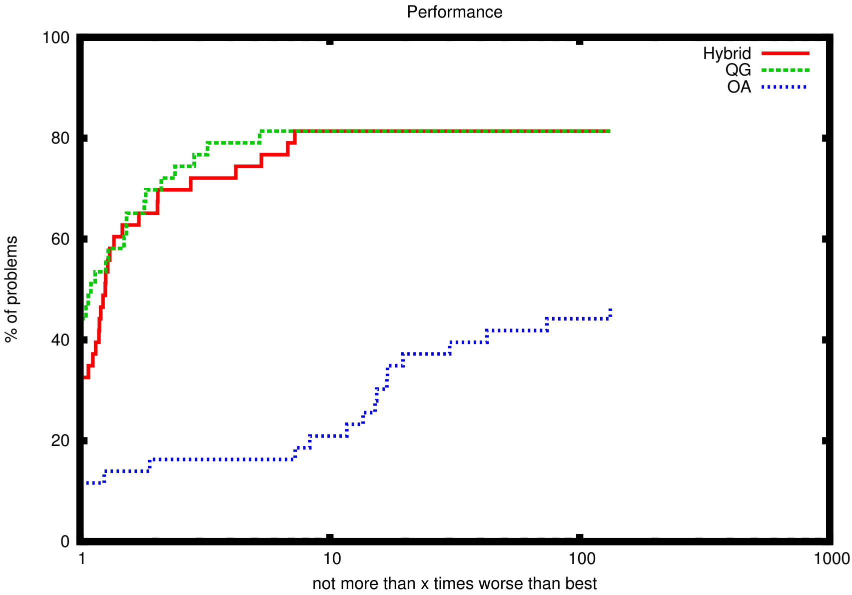
- Test problems
  - Representative selection of 44 convex MINLPs from
    - CMU/IBM library

      http://egon.cheme.cmu.edu/ibm/page.htm
    - MacMinlp [Leyffer]
  - Difficult, but mostly solvable within 3 hour time limit
  - Problem statistics
    - ★ # total vars: 42–1796 (289.8); # discrete vars: 14–432 (93.7)
    - ★ # constraints: 42–3190 (395.4)

Bonmin 0.1.4 with Ipopt (CPU)



Developer Version with FilterSQP (CPU)



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- Variable/node selection
- Primal heuristics
- Presolve
- Cutting planes

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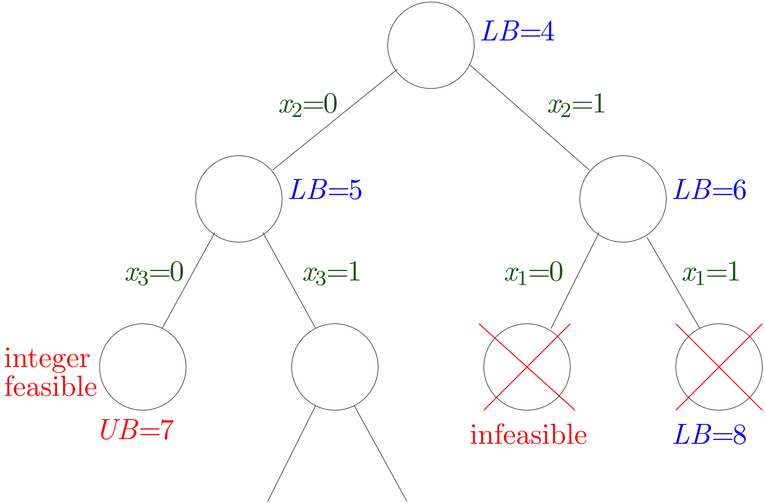
What can we learn from this for a B&B-based method for MINLP?

Branch-and-bound: Variable Selection



Variable Selection

Some possible options:

- Random
- Most-fractional (most integer-infeasible)
  - used in MINLP-BB [Fletcher, Leyffer]

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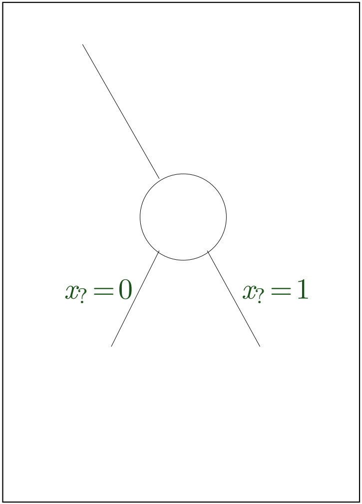
- Random
- Most-fractional (most integer-infeasible)
  - used in `MINLP-BB` [Fletcher, Leyffer]
- Strong branching [Applegate et al. (1995)]
- Pseudo costs [Benichou et al. (1971), Forrest et al. (1974)]
  - optional in `SBB` [GAMS]
- Reliability branching [Achterberg et al. (2005)]

Strong Branching

- Q: Which variable x_i should be branched on?



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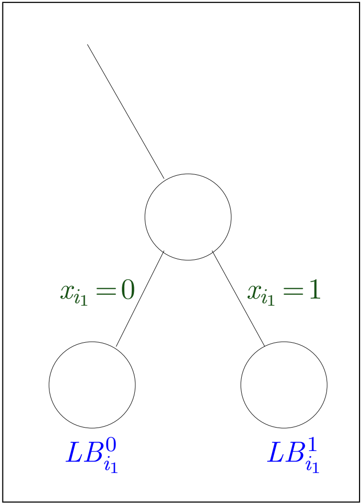

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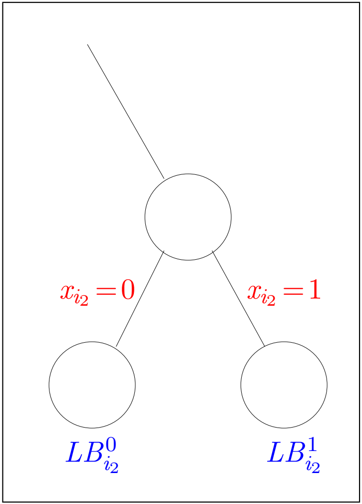

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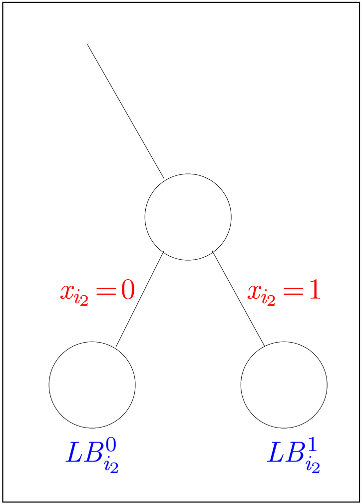

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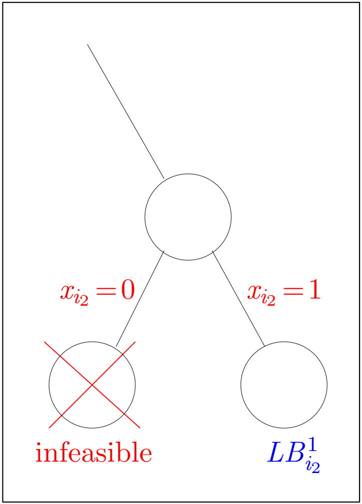

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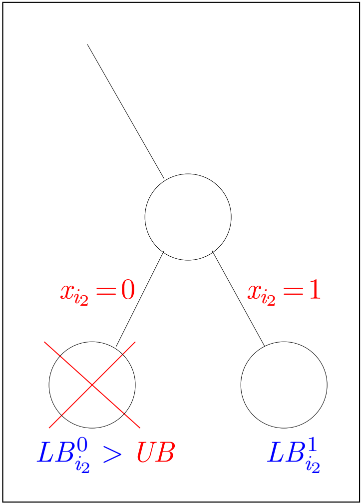

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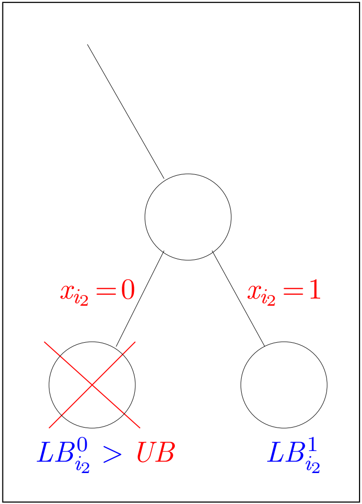
- Requires to solve many relaxations



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- Can use hot-starts (reuse factorization)
  - Only one bound changes

Strong Branching Improvements

Pseudo costs

- Idea: Collect statistical data about the effect of fixing each x_i:
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Reliability branching

- Pseudo costs, but do strong-branching on non-trusted variables
- Limit the number of strong-branching solves

Variable Selection

Comparative experiments in literature:

- MILP
  - Linderoth, Savelsbergh (1999):
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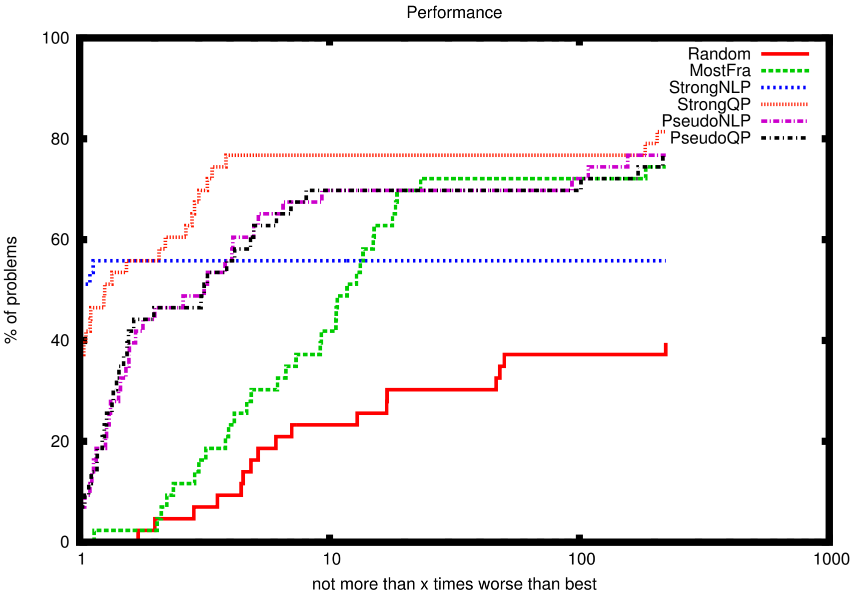
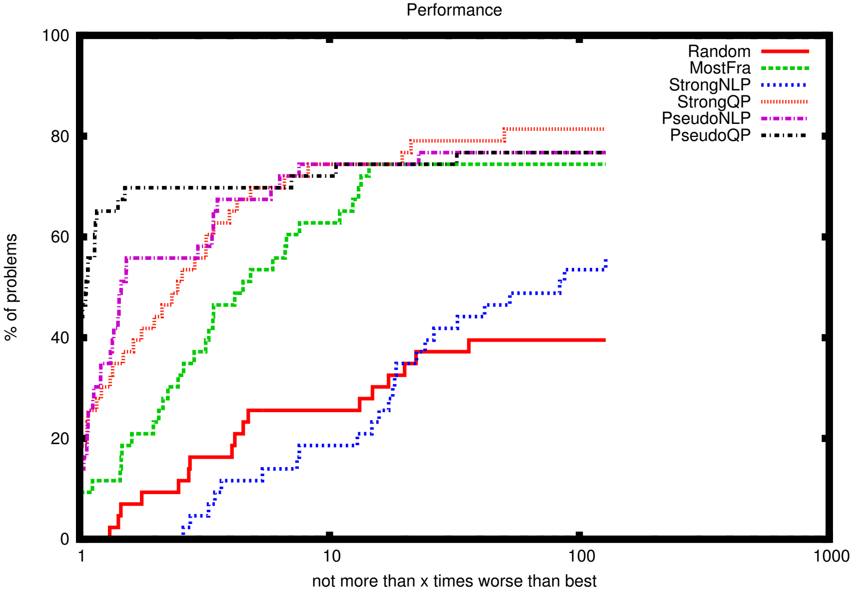
  - Gupta, Ravindran (1985)
    - Most-fractional works best

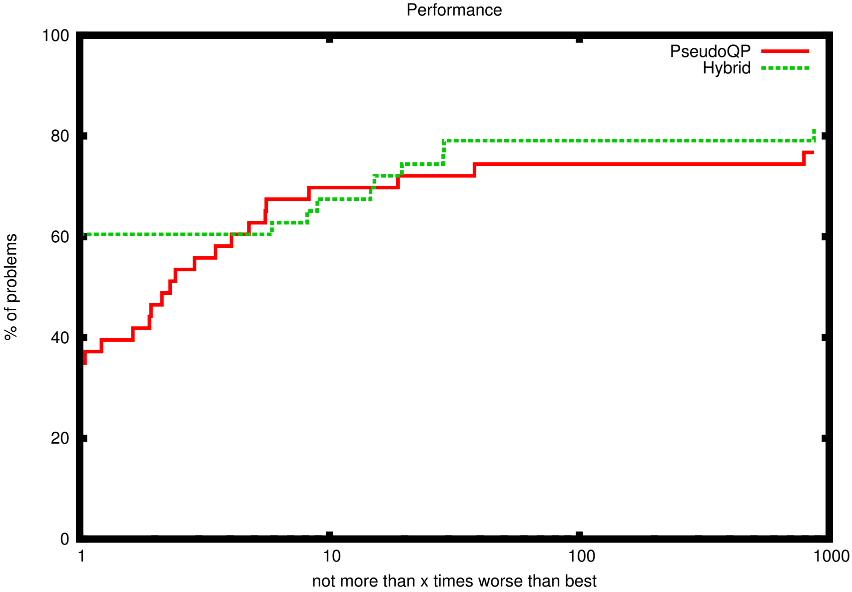
Branch-And-Bound Comparison (# Nodes)



Branch-And-Bound Comparison (CPU time)



B&B and Hybrid Comparison



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- Strong-branching, pseudo-costs work for nonlinear B&B
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- Number of nodes for solved problems:

	Min	Max	GeoMean
Hybrid	8	436393	6226.5
StrongQP	14	2033352	1685.8

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- Need fast detection of infeasibility

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  - Some research specific for nonlinear case:
    - Stubbs, Mehrotra (1999, 2002)
    - Atamtürk, Narayanan (2007)
    - ...
  - Can also use nonlinear cuts
  - Ideally: Need access to problem representation (expression tree)

Other MILP techniques

Primal heuristics (quickly finding good integer feasible points)

- Have answer when time limit exceeded
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Primal heuristics (quickly finding good integer feasible points)

- Have answer when time limit exceeded
- Improve upper bounds (e.g, for strong branching)
- MILP: A dozen generic heuristics (root node and in tree)
  (hot topic over last 7 years)
- MINLP: Preliminary work, e.g.,
  - Nonlinear feasibility pump [Bonami et al. (2006)]

Other MILP techniques

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- In experiments: Use "best-bound" (node with smallest LB)
- Diving
  - Quickly find integer solution
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Presolve (tighten and simplify formulation)

- At root node and in search tree
- MILP: Just look at coefficients of linear functions
- MINLP: General nonlinear functions difficult to predict
  - Requires access to problem representation (e.g., expression tree)

What is Good Modeling?

Example: Uncapacitated facility location problem

$$\begin{array}{rl} \min & \sum_{i=1}^{n} c_i x_i + \sum_{i=1}^{n} \sum_{j=1}^{m} d_{ij} y_{ij} \\ s.t. & \sum_{j=1}^{m} y_{ij} = 1 \quad (i = 1, \ldots, n) \\ \textit{Weak}: & \sum_{i=1}^{n} y_{ij} \leq n \cdot x_i \quad (j = 1, \ldots, m) \\ \textit{Strong}: & y_{ij} \leq x_i \quad (i = 1, \ldots, n;\ j = 1, \ldots, m) \\ & x \in \{0,1\}^n\ ,\ y \in \mathbb{R}_+^m \end{array}$$

	MILP		MINLP	
$n = 30, m = 100$	nodes	time	nodes	time
weak formulation	46,294	143.16		
strong formulation	0	0.18		

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$$
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\min & \sum_{i=1}^{n} c_i x_i + \sum_{i=1}^{n} \sum_{j=1}^{m} d_{ij} y_{ij}^2 \\
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weak with cuts/presolve	25	2.71		

The Nonconvex Case

- Global optimization already very difficult
  - Spatial branch-and-bound with convex under-estimators
  - Incorporation of discrete variables natural
  - Several algorithms and codes:
    `Alpha-BB` [Adjiman et al.], `BARON` [Sahinidis, Tawarmalani],
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  - Nonlinear branch-and-bound
    - resolve NLPs from different starting points
    - do not trust lower bounds or infeasibilities

Conclusions

- Encouraging progress
  - New algorithms and implementations (e.g., `Bonmin`, `FilMINT`)
  - Outer-approximation based algorithms
    - MILP framework with NLP solves
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- Need representative real-world test problems

Thank you!