R. Weismantel

Mixed integer cutting plane theory: a geometric view

Otto-von-Guericke-Universität Magdeburg

Mixed Integer Programming and cutting planes

A mixed integer linear program

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\begin{aligned}
\max \quad & c^T x \\
\text{st.} \quad & Ax = b, \\
& x_i \in \mathbf{Z}_+,\ i \in I,\ x_i \in \mathbf{R}_+,\ i \notin I.
\end{aligned}
$$

The simplex tableau representation

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\begin{aligned}
\max \quad & \bar{c}_N^T x_N \\
\text{st.} \quad & \bar{A}_N x_N \le \bar{b}, \\
& x_i \in \mathbf{R}_+,\ x_i \in \mathbf{Z}_+,\ i \in I.
\end{aligned}
$$

and,

$$
\begin{aligned}
N_i^+ &= \{j \in N \mid \bar{a}_{ij} \ge 0\}, \\
N_i^- &= \{j \in N \mid \bar{a}_{ij} < 0\}, \\
f(g) &= (g_j - \lfloor g_j \rfloor) \text{ for } g \in \mathbf{R}^d.
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A notation

Often the variables are (x, y). Then $x \in \mathbf{Z}^n$ and $y \in \mathbf{R}^d$.

For a closed convex set C, let

$$C_I := \{(x, y) \in C \cap (\mathbf{Z}^n \times \mathbf{R}^d)\}.$$

What is a cutting plane?

For $S \subseteq \mathbf{Z}^n \times \mathbf{R}^d$ a linear inequality $a^T(x, y) \le b$ is valid, i.e.,

$$S \subseteq \{(x, y) \mid a^T(x, y) \le b\}$$

and cuts off the LP optimum (x^*, y^*), i.e, $a^T(x^*, y^*) > b$.

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The fundamental questions in mixed integer cutting plane theory

The point of departure

For a polyhedron $P \subseteq \mathbf{R}^{n+d}$, the set $S = \text{conv}(P \cap (Z^n \times \mathbf{R}^d))$ is a polyhedron. Hence,

$$c^* = \max c^T x + g^T y \quad = \quad \max c^T x + g^T y$$
$$(x,y) \in P \cap \left(Z^n \times \mathbf{R}^d\right) \qquad (x,y) \in S.$$

(1) Which geometric tools are needed in order to understand S?

(2) Which algebraic tools are needed to generate cutting planes?

(3) Can (1) and (2) be turnt into a finite algorithm that computes c^*?

The pure integer setting

- Rounding of hyperplanes: $\sum_i a_i x_i \leq a_0$ turns into $\sum_i \lfloor a_i \rfloor x_i \leq \lfloor a_0 \rfloor$.
- Cutting plane proofs [Chv 73].
- Finiteness [Gomory 58].

The mixed integer setting

- In the mixed 0-1-case, things are nice.
- In general, nothing extends easily. Why?

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The dilemma for general mixed integer programs

An intriguing small example [Cook, Kannan, Schrijver 90]

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\max & & & +y & \\
 & -x_1 & & +y & \leq 0 \\
 & & -x_2 & +y & \leq 0 \\
 & +x_1 & +x_2 & +y & \leq 2 \\
 & x_1 \in \mathbf{Z}_+, & x_2 \in \mathbf{Z}_+, & y \geq 0. &
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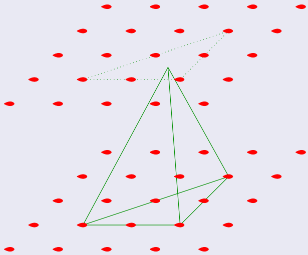
with fractional optimum $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$.



In understanding this example . . ., we need

For $\left\{(x, y) \in S \subset \mathbf{R}^{n+d}\right\}$, its projection is

$$\mathrm{proj}_x(S) = \{x \in \mathbf{R}^n \mid \exists y \text{ such that } (x, y) \in S\}.$$

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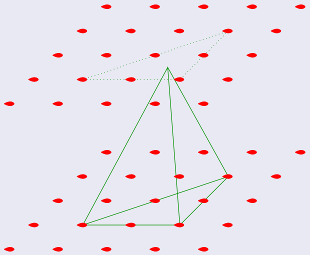

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Ingredients

- $z \in \{0,1\}$ *iff* $z^2 = z$.
- For polyhedra

$$P_1, P_2 \subseteq \mathbf{R}^n,$$

conv$(P_1 \cup P_2)$ can be compactly described. [Balas 79,85].

The system

$$S = \{x \in \{0,1\}^n \times \mathbf{R}^d_+ : Ax \le b\}.$$

The Lift-and-Project Algorithm

- Step 1: Select $j \in N$.
- Step 2: Generate the nonlinear system Q_j,

$$x_j(Ax-b) \le 0, \; (1-x_j)(Ax-b) \le 0.$$

.

- Step 3: Linearize Q_j by $y_i := x_i x_j$ for $i \neq j$ and $x_j^2 = x_j$.
- Step 4: Project the linearized system onto x-space.

Theorem [Balas, Ceria, Cornuejols 93]

$$\text{proj}_x(Q_j) = \text{conv}\{\{Ax \le b, x_j = 0\} \cup \{Ax \le b, x_j = 1\}\}.$$

$$\text{conv}(S) = (\ldots(\text{proj}_x(Q_1))_2 \ldots)_n.$$

Similar approaches: [Lovasz, Schrijver 91] [Sherali, Adams 90]

The mixed $0-1$-case and disjunctive programming

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The basic model

$$X = \{(x, y) \in \mathbf{Z} \times \mathbf{R}_+ | x + y \geq b\}$$

The only missing inequality is

$$x + \frac{1}{1 - f(b)} y \geq \lceil b \rceil$$

Mixed integer rounding can be applied to general models by aggregating variables,

$$z := \sum_{i \in S} g_i x_i .$$

Lemma. The Gomory fractional cut is obtained from mixed integer rounding.

$$\sum_{j \in N \cap I} \min\left\{ f(\overline{a}_{ij}), \frac{f(\overline{b}_i)(1 - f(\overline{a}_{ij}))}{1 - f(\overline{b}_i)} \right\} x_j + \sum_{j \in N_i^+ \setminus I} \overline{a}_{ij} x_j - \sum_{j \in N_i^- \setminus I} \frac{f(\overline{b}_i) \overline{a}_{ij}}{1 - f(\overline{b}_i)} x_j \geq f(\overline{b}_i).$$

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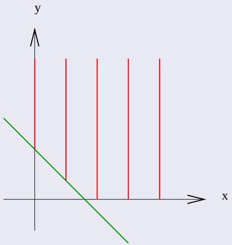

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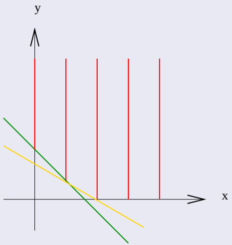

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Mixed integer rounding: a basic one-row-principle [Nemhauser, Wolsey 88]

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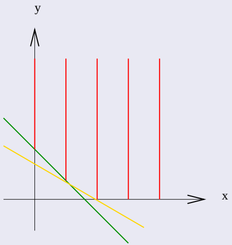

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Cuts from two or more rows of a simplex tableau

A basic model for two and more row -relaxations:

$$f + C_I = \left\{ (x,s) \mid x = f + \sum_{j=1}^{n} \mathbf{r}^j s_j,\ x \in \mathbf{Z}^d,\ s \in \mathbb{Q}_+^n \right\}.$$

The structure of valid inequalities of $f + C_I$ [Andersen, Louveaux, W, Wolsey 06]

A non trivial inequality is of the kind $\sum_{j \in N} \alpha_j s_j \geq 1$ where $\alpha_j \geq 0$.

The coefficients α_j are the recipocal of the distance from f along r^j to the boundary of the projected facet body $\text{proj}_x \left(\{(x,s) \in f + C,\ \alpha^T s = 1\} \right)$.

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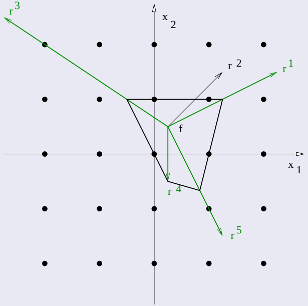
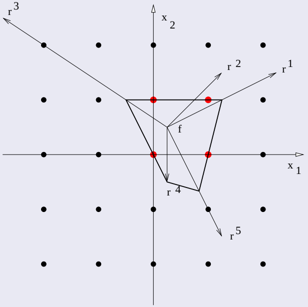
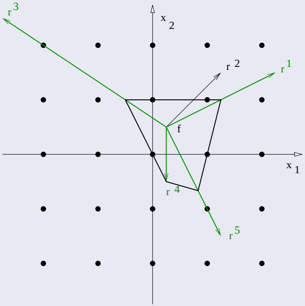


The special case of two rows is geometrically tractable

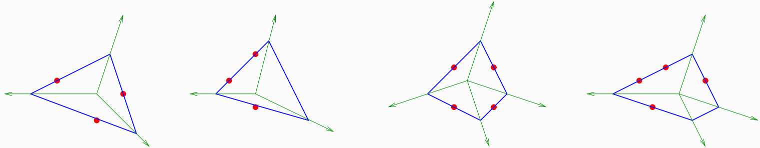
Theorem [Andersen, Louveaux, W, Wolsey 06]



The projected facet body contains no interior integer points. Either 3 or 4 rays (integer points) determine its vertices (boundary).



Classification of the facets by lattice point free polyhedra



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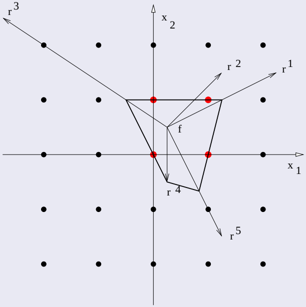

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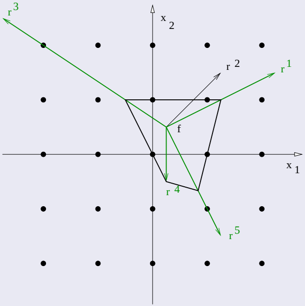

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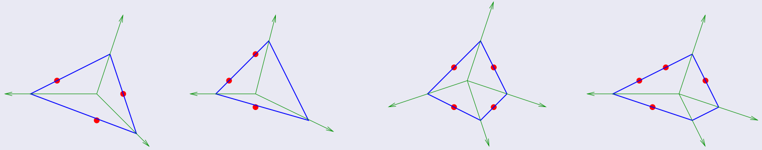

Split cuts [Cook, Kannan, Schrijver 1990]

The algebra

Based on a disjunction

$$\pi^T x \leq \pi_0 \quad \text{or} \quad \pi^T x \geq \pi_0 + 1$$

is valid for $x \in \mathbf{Z}^n$ when π, π_0 are integer.

The geometry

Split cuts [Cook, Kannan, Schrijver 1990]

The algebra

Based on a disjunction

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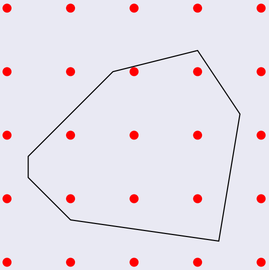

Split cuts [Cook, Kannan, Schrijver 1990]

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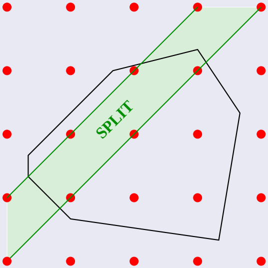

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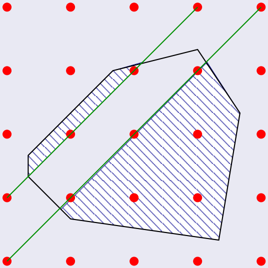

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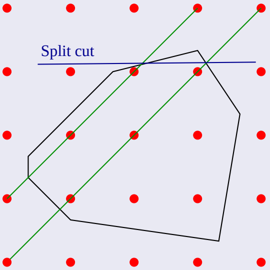

Lattice-point-free polyhedron

A polyhedron P is lattice-point-free when there is no integer point in its interior.

Splits and lpf-polyhedra

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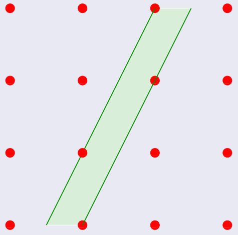

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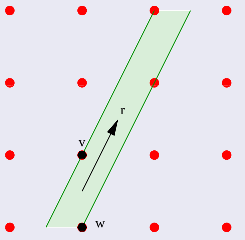

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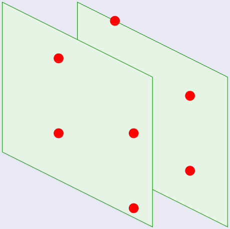

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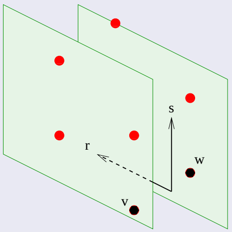

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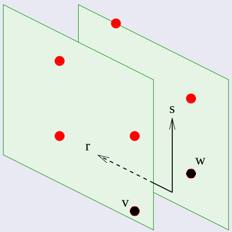

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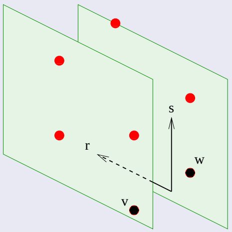

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A generalization of splits based on multi-term disjunctions

Definition

- Let $d^1, \dots, d^k \in \mathbb{Z}^n$ and $\delta_1, \dots, \delta_k \in \mathbb{Z}$. The family $D(k, d, \delta)$ is a *k*-disjunction if for all $x \in \mathbb{Z}^n$ there exists i such that $d^i x \le \delta_i$.
- Let $P \subset \mathbb{R}^n$ be a polyhedron and $c^T x \le \gamma$ be valid for P_I. Then $c^T x \le \gamma$ is a *k*-disjunctive cut for P_I, if there exists a k-disjunction $D(k, d, \delta)$ with

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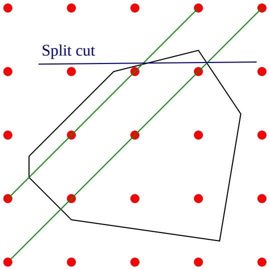

Proposition

Let $P \subseteq \mathbb{R}^{n+d}$ be a polyhedron. Every valid inequality $c^T x \le \gamma$ for P_I is a 2^n-disjunctive cut for some k.

Theorem [Jörg 07]

There is a finite cutting plane algorithm for any bounded mixed integer program based on k-disjunctions.

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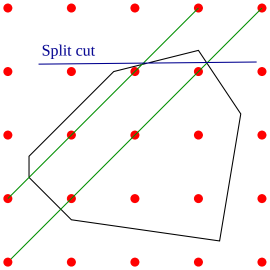

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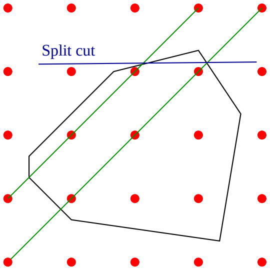

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- A lpf polyhedron $\text{conv}\{v^1, \ldots, v^p\} + \text{span}\{w^1, \ldots, w^q\}$ is called split body, the number $n - q$ the split-dimension.

Examples

A lpf triangle in $\mathbf{R}^2$ has split dimension two.

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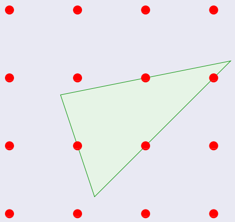

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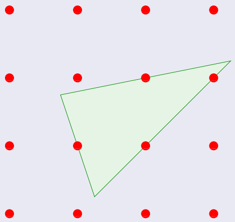

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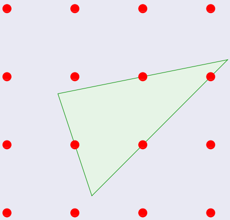

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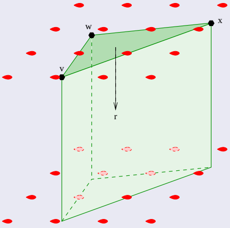

Split bodies give rise to cuts for convex mixed integer programs.

An operation

For a split body $L \subseteq \mathbf{R}^n$ and a closed convex set C, let

$$R(L) := \text{ cl } \operatorname{conv}(\{(x, y) \in C : x \notin \operatorname{rint}(L)\}).$$

Then, $\operatorname{conv}(C_I) \subseteq R(L) \subseteq C$.

Lemma [Andersen, Louveaux, W 07]

Let $L \subseteq \mathbf{R}^n$ be a split body.

- For a closed convex set C, $R(L) \neq C$ iff there exists an extreme point (x, y) of C such that x is in the interior of L.
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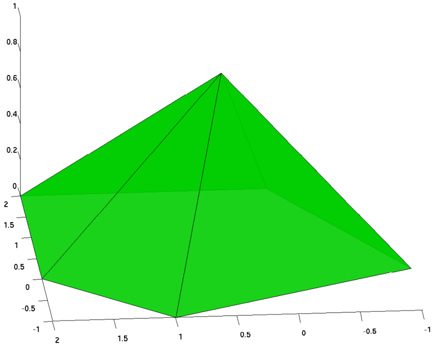

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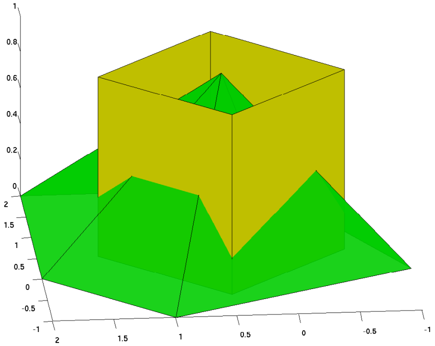

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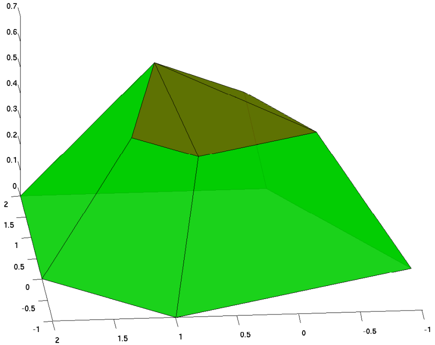

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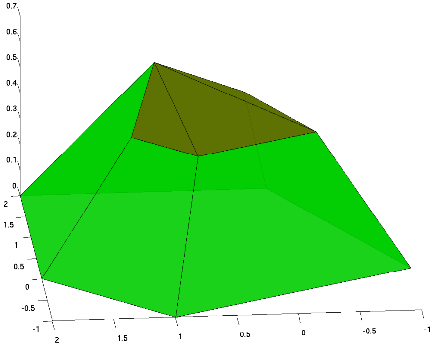

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From split bodies to cutting plane proofs

The closure of split bodies

For a family $\mathcal{F}$ of split bodies, let

$$\mathrm{Cl}(\mathcal{F}, C) := \bigcap_{L \in \mathcal{F}} R(L).$$

Letting $C^0(\mathcal{F}, C) = C$, define for $i \geq 1$,

$$C^i(\mathcal{F}, C) = \mathrm{Cl}(\mathcal{F}, C^{i-1}(\mathcal{F}, C)).$$

If $Ax \leq b$ is full dimensional and lpf, then $y \leq 0$ has split size n w.r.t. $Ax + \mathbf{1}y \leq b,\ x \in \mathbf{Z}^n,\ y \geq 0$.

Theorem [Cook, Kannan, Schrijver 90]

For a polyhedron P and the set $\mathcal{F}$ of split bodies of split dimension one, $\mathrm{Cl}(\mathcal{F}, P)$ is a polyhedron.

Definition

For an inequality $c^T x \leq \gamma$, valid for $\mathrm{conv}(C_I)$, a split body proof is a finite family $\mathcal{F}$ of split bodies such that $c^T x \leq \gamma$ is valid for $C^k(\mathcal{F}, C)$ for some k. The split size of the proof is the largest split dimension of a split body in $\mathcal{F}$.

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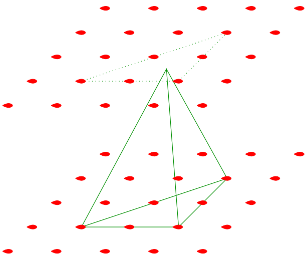

Definition

For an inequality $c^T x \leq \gamma$, valid for $\mathrm{conv}(C_I)$, a split body proof is a finite family $\mathcal{F}$ of split bodies such that $c^T x \leq \gamma$ is valid for $C^k(\mathcal{F}, C)$ for some k. The split size of the proof is the largest split dimension of a split body in $\mathcal{F}$.

Split bodies: a perspective

Theorem [Andersen, Louveaux, W 07]

Let $c^T x \leq \gamma$ be a nt valid inequality for $\text{conv}(C_I)$.

(i) It has a split body proof of split size split-dim(c, γ).

(ii) There is no split body proof of split size smaller than split-dim(c, γ).

Corollary

Any cutting plane algorithm based on split bodies of split-dimension less than split-dim(c, γ) cannot solve the optimization problem $\gamma = \{\max c^T x,\ x \in C_I\}$ in finitely many rounds.

Theorem [Andersen, Louveaux, W 07]

Let F be the optimal face of $\max c^T x \mid x \in P$. F contains no mixed integer points iff there exists a split body of split size at most $\max\{1, \dim F\}$ containing F in its interior.

How to find split bodies?

Let (x^*, y^*) be an optimal vertex for $\max c^T x + d^T y :\ Ax + By \leq b$. Let I be the tight rows at (x^*, y^*) with $[A, C]_I$ of rank $n + d$. Then,

$$\mathcal{L}^* = \{z \in \mathbb{Q}^I \mid z^T A_I \in \mathbf{Z}^n, \quad z^T C_I = 0\}$$

is a lattice. Any basis $\{z_1, \ldots, z_k\}$ of $\mathcal{L}^*$ satisfies $b^T z_i \in \mathbf{Z}$ for all i iff $x^* \in \mathbf{Z}^n$.

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