Stochastic Optimization for Big Data Analytics: Algorithms and Libraries

Tianbao Yang[‡]
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collaborators:
Rong Jin[†], Shenghuo Zhu[‡]
[‡]NEC Laboratories America, [†]Michigan State University

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The updates are available here

http://www.cse.msu.edu/~yangtia1/sdm14-tutorial.pdf

Thanks.

Outline

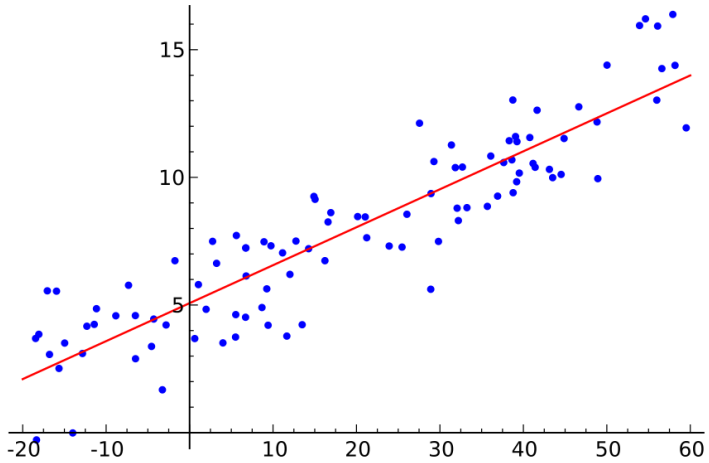
Introduction

- Machine Learning problems and Stochastic Optimization
- Coverage:
  - Classification and Regression in different forms
  - Motivation to employ STochastic OPtimization (STOP)
  - Basic Convex Optimization Knowledge

Learning as Optimization

Least Square Regression Problem:

$$\min_{\mathbf{w}\in\mathbb{R}^d} \frac{1}{n}\sum_{i=1}^{n}(y_i - \mathbf{w}^\top \mathbf{x}_i)^2 + \frac{\lambda}{2}\|\mathbf{w}\|_2^2$$

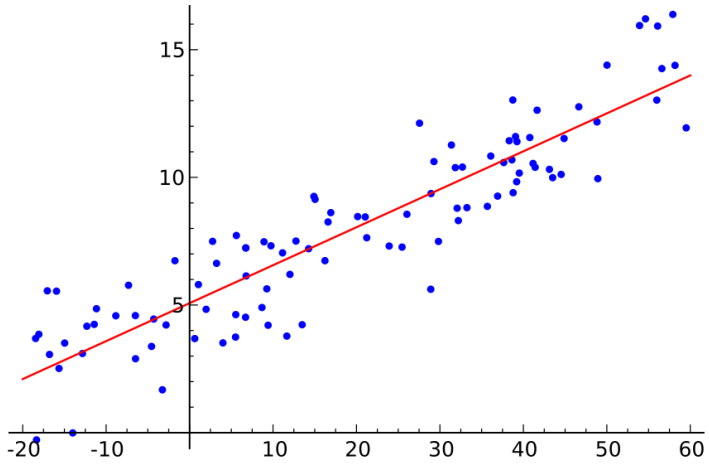


- $\mathbf{x}_i \in \mathbb{R}^d$: d-dimensional feature vector
- $y_i \in \mathbb{R}$: response variable
- $\mathbf{w} \in \mathbb{R}^d$: unknown parameters
- n: number of data points

Learning as Optimization

Least Square Regression Problem:

$$\min_{\mathbf{w}\in\mathbb{R}^d} \underbrace{\frac{1}{n}\sum_{i=1}^{n}(y_i - \mathbf{w}^\top \mathbf{x}_i)^2}_{\text{Empirical Loss}} + \frac{\lambda}{2}\|\mathbf{w}\|_2^2$$



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Learning as Optimization

Least Square Regression Problem:

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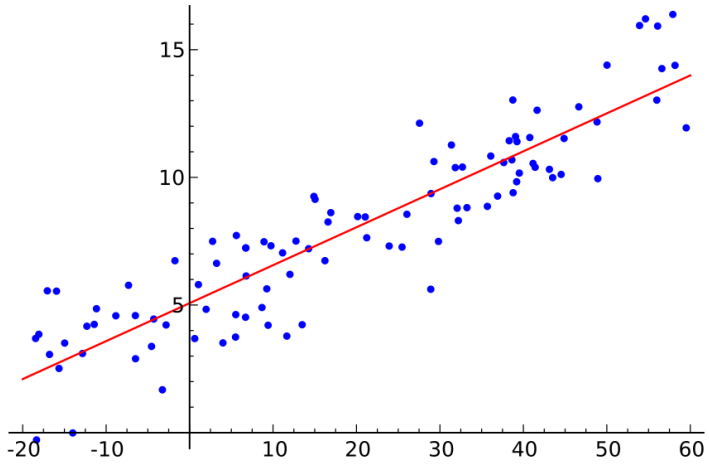

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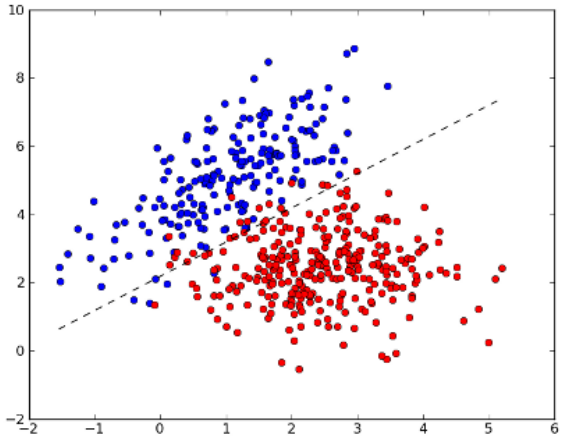
Learning as Optimization

Classification Problems:

$$\min_{\mathbf{w}\in\mathbb{R}^d} \frac{1}{n}\sum_{i=1}^{n} \ell(y_i \mathbf{w}^\top \mathbf{x}_i) + \frac{\lambda}{2}\|\mathbf{w}\|_2^2$$

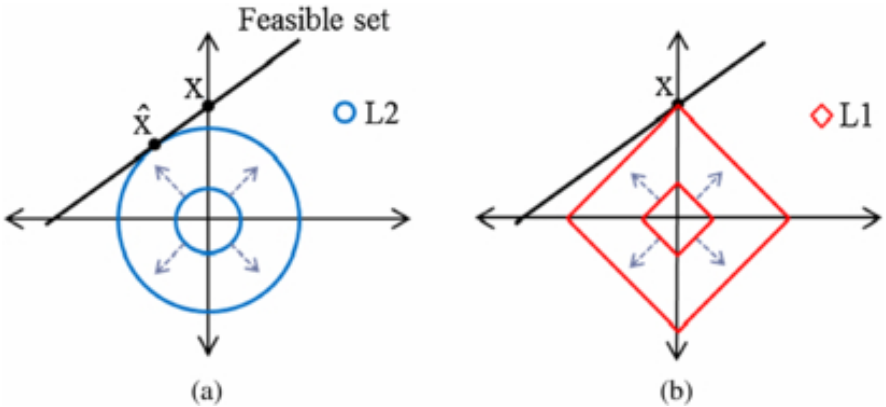


- $y_i \in \{+1, -1\}$: label
- Loss function $\ell(z)$
  1. SVMs: hinge loss $\ell(z) = \max(0, 1-z)^p$, where $p = 1, 2$
  2. Logistic Regression: $\ell(z) = \log(1 + \exp(-z))$

Learning as Optimization

Feature Selection:

$$\min_{\mathbf{w}\in\mathbb{R}^d} \frac{1}{n}\sum_{i=1}^{n} \ell(y_i \mathbf{w}^\top \mathbf{x}_i) + \lambda \|\mathbf{w}\|_1$$

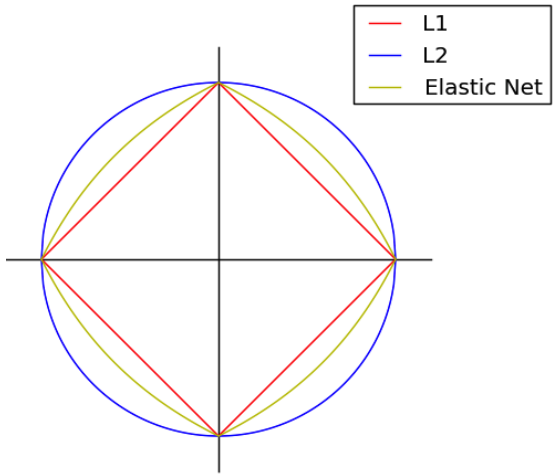


- ℓ_1 regularization $\|\mathbf{w}\|_1 = \sum_{i=1}^{d} |w_i|$
- λ controls sparsity level

Learning as Optimization

Feature Selection using Elastic Net:

$$\min_{\mathbf{w}\in\mathbb{R}^d} \frac{1}{n}\sum_{i=1}^{n} \ell(y_i \mathbf{w}^\top \mathbf{x}_i) + \lambda \left(\|\mathbf{w}\|_1 + \gamma \|\mathbf{w}\|_2^2 \right)$$

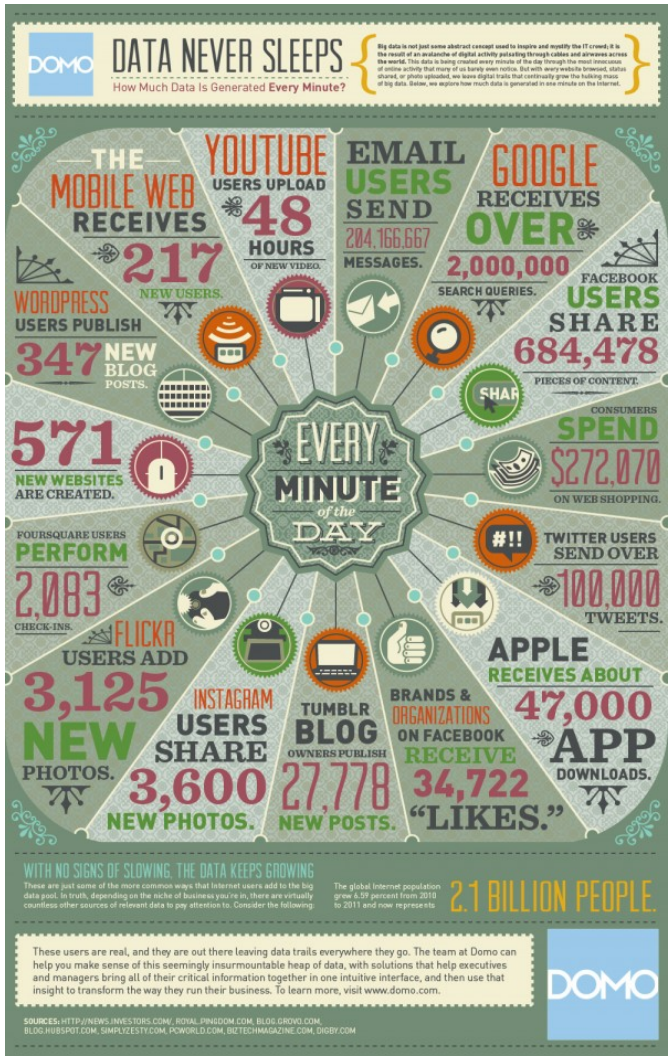


- Elastic net regularizer, more robust than ℓ_1 regularizer

Big Data Challenge

Huge amount of data generated on the internet every day

- Facebook users upload 3 million photos
- Goolge receives 3000 million queries
- GE 100 Million Hours of operational data on the World's Largest Gas Turbine Fleet
- The global internet population is 2.1 billion people



http://www.visualnews.com/2012/06/19/how-much-data-created-every-minute/

Why Learning from Big Data is Hard?

$$\min_{\mathbf{w}\in\mathbb{R}^d} \underbrace{\frac{1}{n}\sum_{i=1}^{n} \ell(\mathbf{w}^\top \mathbf{x}_i, y_i) + \lambda R(\mathbf{w})}_{\text{empirical loss + regularizer}}$$

Too many data points

- Issue: can't afford go through data set many times
- Solution: Stochastic Optimization

Why Learning from Big Data is Hard?

$$\min_{\mathbf{w}\in\mathbb{R}^d} \underbrace{\frac{1}{n}\sum_{i=1}^{n}\ell(\mathbf{w}^\top\mathbf{x}_i, y_i) + \lambda R(\mathbf{w})}_{\text{empirical loss + regularizer}}$$

High Dimensional Data

- Issue: can't afford second order optimization (e.g. Newton's method)
- Solution: first order method (i.e, gradient based method)

Why Learning from Big Data is Hard?

$$\min_{\mathbf{w}\in\mathbb{R}^d} \underbrace{\frac{1}{n}\sum_{i=1}^{n} \ell(\mathbf{w}^\top \mathbf{x}_i, y_i) + \lambda R(\mathbf{w})}_{\text{empirical loss + regularizer}}$$

Data are distributed over many machines

- Issue: expensive (if not impossible) to move data
- Solution: Distributed Optimization

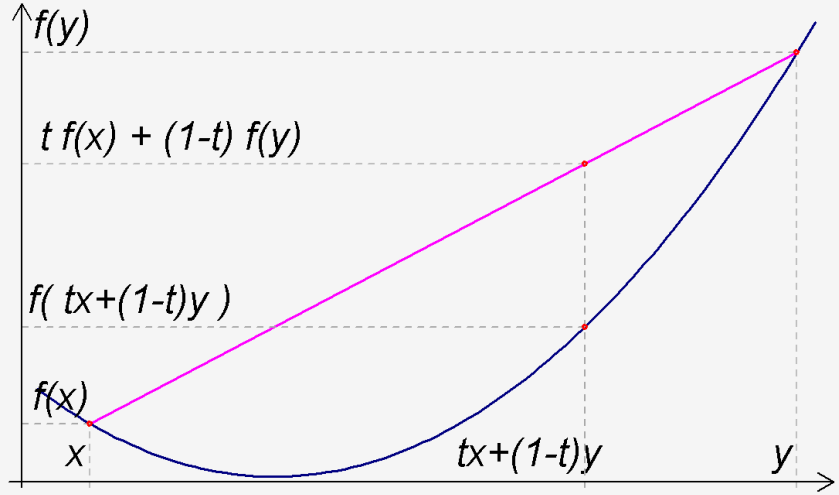
Warm-up: Vector, Norm, Inner product, Dual Norm

- bold letters $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{w} \in \mathbb{R}^d$: d-dimensional vectors
- norm $\|\mathbf{x}\|$: $\mathbb{R}^d \to \mathbb{R}_+$. e.g. ℓ_2 norm $\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^d x_i^2}$, ℓ_1 norm $\|\mathbf{x}\|_1 = \sum_{i=1}^d |x_i|$, ℓ_∞ norm $\|\mathbf{x}\|_\infty = \max_i |x_i|$
- inner product $< \mathbf{x}, \mathbf{y} >= \mathbf{x}^\top \mathbf{y} = \sum_{i=1}^d x_i y_i$
- dual norm $\|\mathbf{x}\|_* = \max_{\|\mathbf{y}\| \leq 1} \mathbf{x}^\top \mathbf{y}$. $\|\mathbf{x}\|_2 \Longleftrightarrow \|\mathbf{x}\|_2$, $\|\mathbf{x}\|_1 \Longleftrightarrow \|\mathbf{x}\|_\infty$

Warm-up: Convex Optimization

$$\min_{\mathbf{x}\in\mathcal{X}} f(\mathbf{x})$$

- $f(\mathbf{x})$ is a convex function
- $\mathcal{X}$ is a convex domain



$$f(\alpha\mathbf{x}+(1-\alpha)\mathbf{y}) \leq \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y}), \forall \mathbf{x}, \mathbf{y} \in \mathcal{X}, \alpha \in [0,1]$$

Warm-up: Convergence Measure

- Most optimization algorithms are iterative

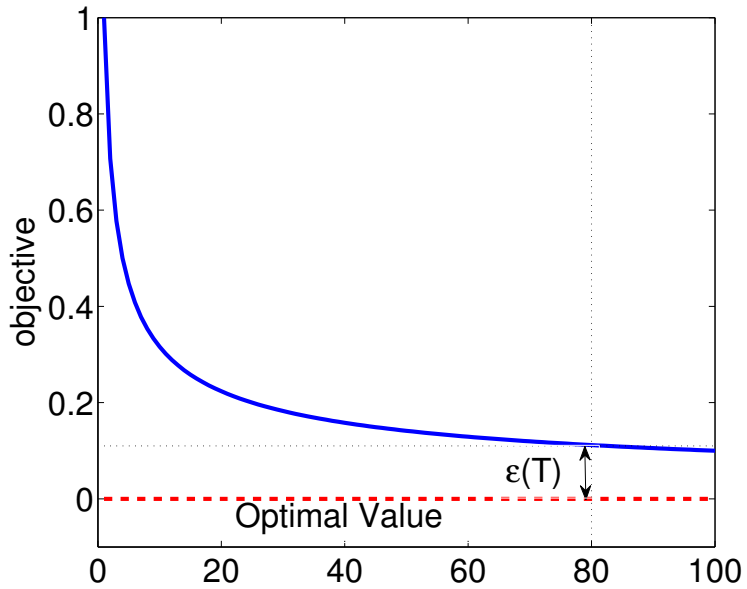
$$\mathbf{w}_{t+1} = \mathbf{w}_t + \Delta \mathbf{w}_t$$

- Iteration Complexity: the number of iterations $T(\epsilon)$ needed to have

$$f(\mathbf{x}_T) - \min_{\mathbf{x} \in \mathcal{X}} f(\mathbf{x}) \leq \epsilon \quad (\epsilon \ll 1)$$

- Convergence rate: after T iterations, how good is the solution

$$f(\mathbf{x}_T) - \min_{\mathbf{x} \in \mathcal{X}} f(\mathbf{x}) \leq \epsilon(T)$$



More on Convergence Measure

	Convergence Rate	Iteration Complexity
linear	$\mu^T \quad (\mu < 1)$	$O\left(\log(\frac{1}{\epsilon})\right)$
sub-linear	$\frac{1}{T^\alpha} \quad \alpha > 0$	$O\left(\frac{1}{\epsilon^{1/\alpha}}\right)$

Why are we interested in Bounds?

More on Convergence Measure

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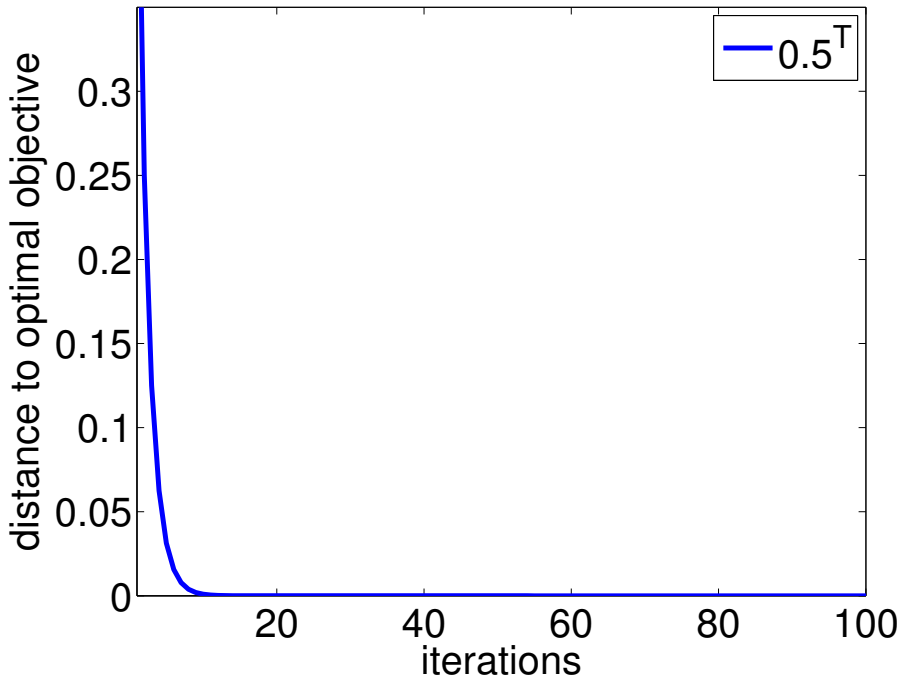

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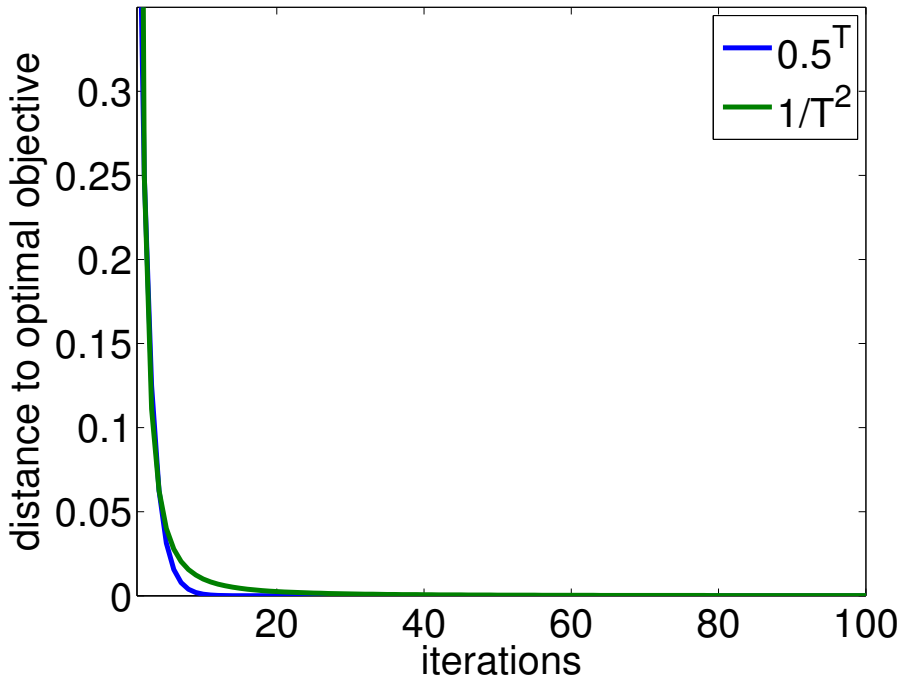

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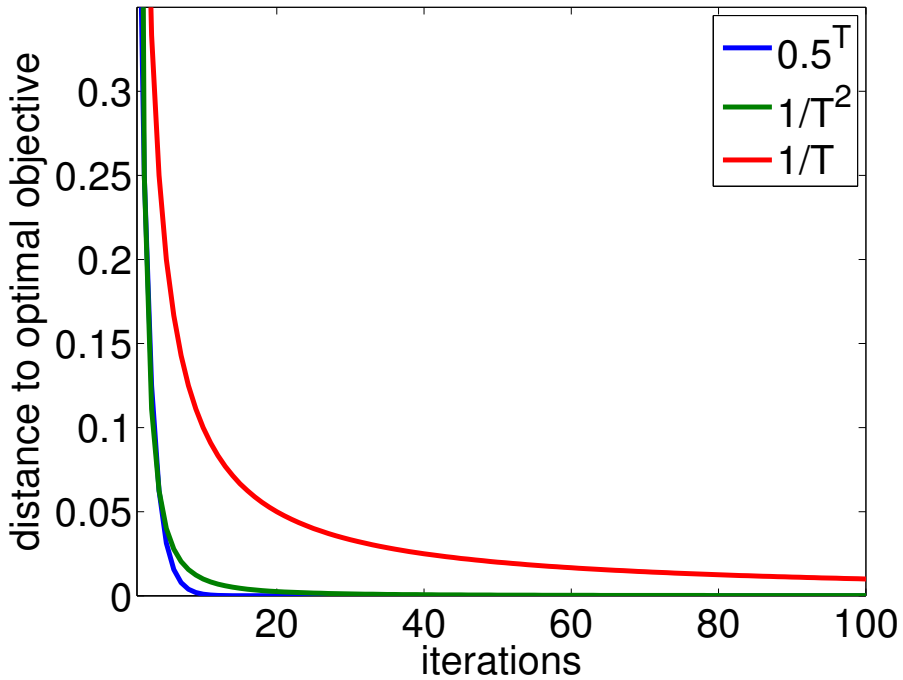

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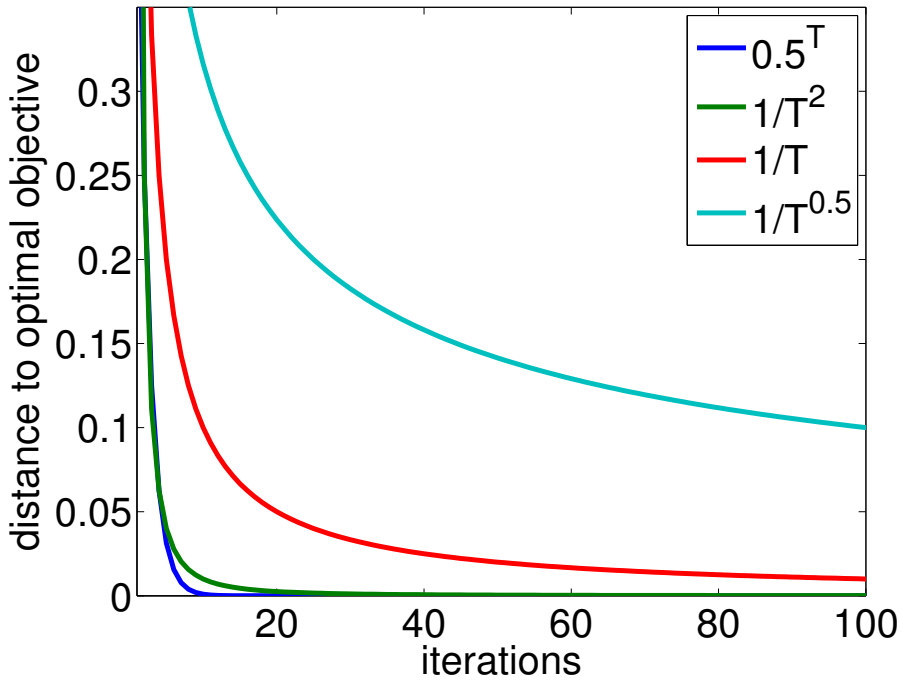

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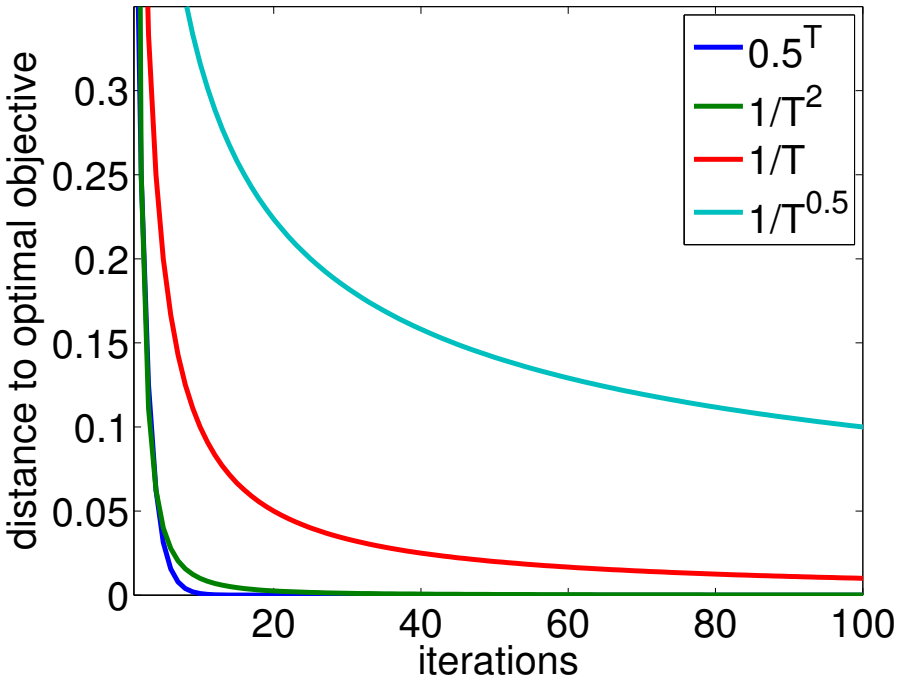


$$\mu^T < \frac{1}{T^2} < \frac{1}{T} < \frac{1}{\sqrt{T}}$$

$$\log\left(\frac{1}{\epsilon}\right) < \frac{1}{\sqrt{\epsilon}} < \frac{1}{\epsilon} < \frac{1}{\epsilon^2}$$

Factors that affect Iteration Complexity

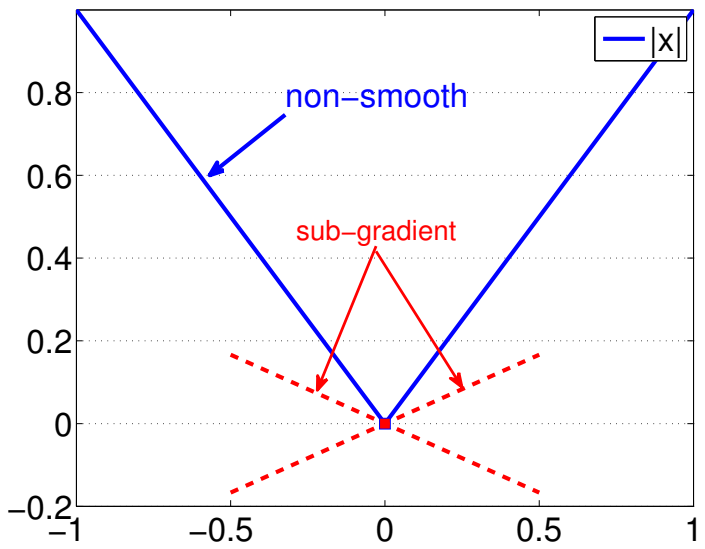
- Domain $\mathcal{X}$: size and geometry
- Size of problem: dimension and number of data points
- Property of function: smoothness of function

Warm-up: Non-smooth function

- Lipschitz continuous: e.g. absolute

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq L\|\mathbf{x} - \mathbf{y}\|$$

$$f(\mathbf{x}) \geq f(\mathbf{y}) + \partial f(\mathbf{y})^\top (\mathbf{x} - \mathbf{y})$$



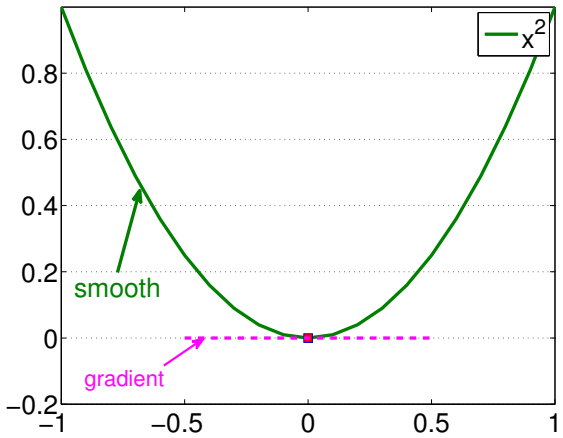
- Iteration complexity $O\left(\frac{1}{\epsilon^2}\right)$ or convergence rate $O\left(\frac{1}{\sqrt{T}}\right)$

Warm-up: Smooth Convex function

- smooth: e.g. logistic loss $f(z) = \log(1 + \exp(-z))$

$$\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\|_* \leq \beta \|\mathbf{x} - \mathbf{y}\|$$

where $\beta > 0$



Second Order Derivative is upper bounded

- Full gradient descent $O\left(\frac{1}{\sqrt{\epsilon}}\right)$ or $O\left(\frac{1}{T^2}\right)$
- Stochastic gradient descent $O\left(\frac{1}{\epsilon^2}\right)$ or $O\left(\frac{1}{\sqrt{T}}\right)$

Warm-up: Strongly Convex function

- strongly convex: e.g. ℓ_2^2 norm $f(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|_2^2$

$$\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\|_* \geq \lambda \|\mathbf{x} - \mathbf{y}\|, \quad \lambda > 0$$

- Second Order Derivative is lower bounded
- Iteration complexity $O(1/\epsilon)$ or convergence rate $O(1/T)$

Warm-up: Smooth and Strongly Convex function

- smooth and strongly convex:

$$\lambda\|\mathbf{x}-\mathbf{y}\| \leq \|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\|_* \leq \beta\|\mathbf{x}-\mathbf{y}\|, \quad \beta \geq \lambda > 0$$

- e.g. square loss: $f(z) = \frac{1}{2}(1-z)^2$
- Iteration complexity $O\left(\log\left(\frac{1}{\epsilon}\right)\right)$ or convergence rate $O(\mu^T)$

Warm-up: Stochastic Gradient

$$\text{Loss: } L(\mathbf{w}) = \frac{1}{n}\sum_{i=1}^{n} \ell(\mathbf{w}^\top \mathbf{x}_i, y_i)$$

- stochastic gradient: $\nabla \ell(\mathbf{w}^\top \mathbf{x}_{i_t}, y_{i_t})$
- where $i_t \in \{1, \ldots, n\}$ randomly sampled
- key equation: $\mathrm{E}_{i_t}[\nabla \ell(\mathbf{w}^\top \mathbf{x}_{i_t}, y_{i_t})] = \nabla L(\mathbf{w})$
- computation is very cheap

Warm-up: Dual

Convex Conjugate: $\ell^*(\alpha) \Longleftrightarrow \ell(\mathbf{z})$

$$\ell(z) = \max_{\alpha \in \Omega} \alpha z - \ell^*(\alpha), \quad \textit{e.g.}\, \max(0, 1-z) = \max_{\alpha \leq 0} \alpha z - \alpha$$

Dual Objective (classification):

$$\min_{\mathbf{w}} \frac{1}{n} \sum_{i=1}^{n} \ell(\underbrace{y_i \mathbf{w}^\top \mathbf{x}_i}_{z}) + \frac{\lambda}{2} \|\mathbf{w}\|_2^2$$

$$= \min_{\mathbf{w}} \frac{1}{n} \sum_{i=1}^{n} \left[\max_{\alpha_i \in \Omega} \alpha_i (y_i \mathbf{w}^\top \mathbf{x}_i) - \ell^*(\alpha_i) \right] + \frac{\lambda}{2} \|\mathbf{w}\|_2^2$$

$$= \max_{\alpha \in \Omega} \frac{1}{n} \sum_{i=1}^{n} -\ell^*(\alpha_i) - \frac{\lambda}{2} \left\| \frac{1}{\lambda n} \sum_{i=1} \alpha_i y_i \mathbf{x}_i \right\|_2^2$$

Outline

STOP Algorithms for Big Data Classification and Regression

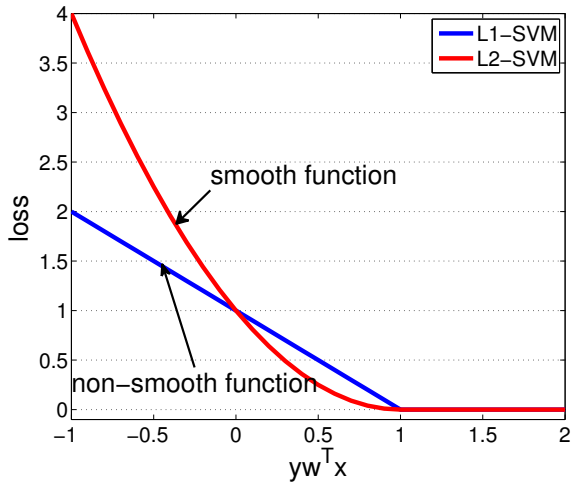
In this section, we present several well-known STOP algorithms for solving classification and regression problems

- Coverage:
  - Stochastic Gradient Descent (Pegasos) for L1-SVM (primal)
  - Stochastic Dual Coordinate Ascent (SDCA) for L2-SVM (dual)
  - Stochastic Average Gradient (SAG) for Logistic Regression/Regression (primal)

STOP Algorithms for Big Data Classification

Support Vector Machines:

$$\min_{\mathbf{w}\in\mathbb{R}^d} f(\mathbf{w}) = \frac{1}{n}\sum_{i=1}^{n} \underbrace{\ell(\mathbf{w}^\top \mathbf{x}_i, y_i)}_{loss} + \frac{\lambda}{2}\|\mathbf{w}\|_2^2$$



- L_1 SVM: $\ell(\mathbf{w}^\top \mathbf{x}, y) = \max(0, 1 - y_i \mathbf{w}^\top \mathbf{x}_i)$
- L_2 SVM: $\ell(\mathbf{w}^\top \mathbf{x}, y) = \max(0, 1 - y_i \mathbf{w}^\top \mathbf{x}_i)^2$
- Equivalent to C-SVM formulations $C = n\lambda$

STOP Algorithms for Big Data Classification

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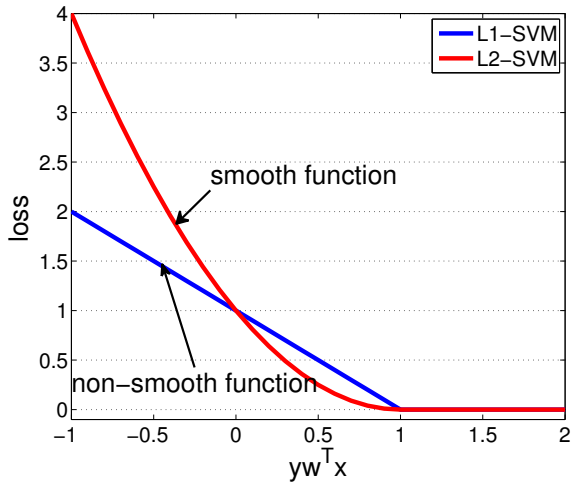

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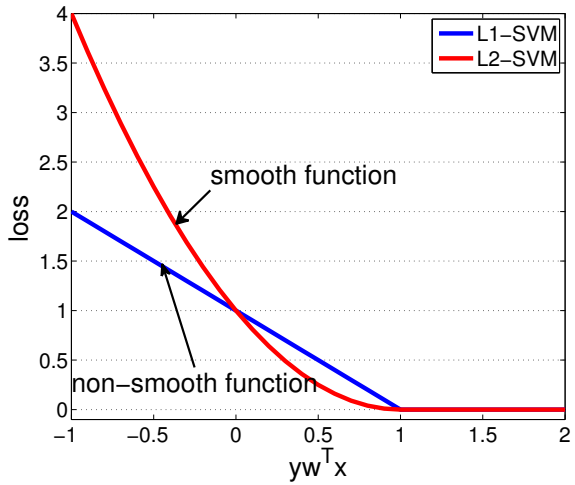

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- L_2 SVM: $\ell(\mathbf{w}^\top \mathbf{x}, y) = \max(0, 1 - y_i \mathbf{w}^\top \mathbf{x}_i)^2$
- Equivalent to C-SVM formulations $C = n\lambda$

Pegasos (Shai et al. ICML '07, primal): STOP for L_1 SVM

$$\min_{\mathbf{w}\in\mathbb{R}^d} f(\mathbf{w}) = \frac{1}{n}\sum_{i=1}^{n} \max(0, 1 - y_i \mathbf{w}^\top \mathbf{x}_i) + \underbrace{\frac{\lambda}{2}\|\mathbf{w}\|_2^2}_{\text{strongly convex}}$$

Pegasos (SGD for strongly convex) :

$$\text{stochastic gradient: } \partial \ell(\mathbf{w}^\top \mathbf{x}_i, y_i) = \begin{cases} -y_i \mathbf{x}_i, & 1 - y_i \mathbf{w}^\top \mathbf{x}_i < 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\partial f(\mathbf{w}_t; i_t) = \partial \ell(\mathbf{w}_t^\top \mathbf{x}_{i_t}, y_{i_t}) + \lambda \mathbf{w}_t$$

Stochastic Gradient Descent Updates

$$\mathbf{w}_{t+1} = \mathbf{w}_t - \frac{1}{\lambda t}\partial f(\mathbf{w}_t; i_t)$$

SDCA (C.J. Hsieh, ICML '08, Shai&Tong JMLR'12, dual): STOP for L_2 SVM

convex conjugate: $\max(0, 1 - y\mathbf{w}^\top \mathbf{x})^2 = \max_{\alpha \geq 0} \left(\alpha - \frac{1}{4}\alpha^2 \right) - \alpha y \mathbf{w}^\top \mathbf{x}$

Dual SVM: $\max_{\alpha \geq 0} D(\alpha) = \frac{1}{n} \sum_{i=1}^{n} \underbrace{\left(\alpha_i - \frac{\alpha_i^2}{4} \right)}_{\phi^*(\alpha_i)} - \frac{\lambda}{2} \left\| \frac{1}{\lambda n} \sum_{i=1}^{n} \alpha_i y_i \mathbf{x}_i \right\|_2^2$

primal solution: $\mathbf{w}_t = \frac{1}{\lambda n} \sum_{i=1}^{n} \alpha_i^t y_i \mathbf{x}_i$

SDCA: stochastic dual coordinate ascent

Dual Coordinate Updates

$$\Delta \alpha_i = \max_{\alpha_i^t + \Delta \alpha_i \geq 0} \phi^*(\alpha_i^t + \Delta \alpha_i) - \frac{\lambda}{2} \left\| \mathbf{w}_t + \frac{1}{\lambda n} \Delta \alpha_i y_i \mathbf{x}_i \right\|_2^2$$

SAG (Schmidt et al. '11, primal): Stopt for Logit Reg./ Reg.

- smooth and strongly convex objective

$$\min_{\mathbf{w}\in\mathbb{R}^d} f(\mathbf{w}) = \underbrace{\frac{1}{n}\sum_{i=1}^{n}\ell(\mathbf{w}^\top \mathbf{x}_i, y_i) + \frac{\lambda}{2}\|\mathbf{w}\|_2^2}_{\text{smooth and strongly convex}}$$

- e.g. logistic regression, least square regression

SAG (stochastic average gradient):

$$\mathbf{w}_{t+1} = (1-\eta\lambda)\mathbf{w}_t - \boxed{\frac{\eta}{n}\sum_{i=1}^{n} g_i^t}, \quad \text{where}$$

$$g_i^t = \begin{cases} \partial\ell(\mathbf{w}_t^\top \mathbf{x}_i, y_i), & \text{if } i \text{ is selected} \\ g_i^{t-1}, & \text{otherwise} \end{cases}$$

OK, tell me which one is the best?

Let us compare Theoretically first

alg.	Pegasos	SAG	SDCA
$f(\mathbf{w})$	sc	sm & sc	sm loss & sc reg.
e.g.	SVMs, LR, LSR	L_2-SVM, LR, LSR	SVMs, LR, LSR
com. cost	$O(d)$	$O(d)$	$O(d)$
mem. cost	$O(d)$	$O(n+d)$	$O(n+d)$
Iteration	$O\left(\frac{1}{\lambda\epsilon}\right)$	$O\left((n+\frac{1}{\lambda})\log\left(\frac{1}{\epsilon}\right)\right)$	$O\left((n+\frac{1}{\lambda})\log\left(\frac{1}{\epsilon}\right)\right)$

Table : sc: strongly convex; sm: smooth; LR: logistic regression; LSR: least square regression

What about ℓ_1 regularization?

$$\min_{\mathbf{w}\in\mathbb{R}^d} \frac{1}{n}\sum_{i=1}^{n} \ell(\mathbf{w}^\top \mathbf{x}_i, y_i) + \sigma \underbrace{\sum_{g=1}^{K} \|\mathbf{w}_g\|_1}_{\text{Lasso or Group Lasso}}$$

Issue: Not Strongly Convex Solution: Add ℓ_2^2 regularization

$$\min_{\mathbf{w}\in\mathbb{R}^d} \frac{1}{n}\sum_{i=1}^{n} \ell(\mathbf{w}^\top \mathbf{x}_i, y_i) + \sigma \sum_{g=1}^{K} \|\mathbf{w}_g\|_1 + \boxed{\frac{\lambda}{2}\|\mathbf{w}\|_2^2}$$

setting $\lambda = \Theta(1/\epsilon)$

Algorithm: Pegasos, SDCA

Outline

General Strategies for Stochastic Optimization

General strategies for Stochastic Gradient Descent

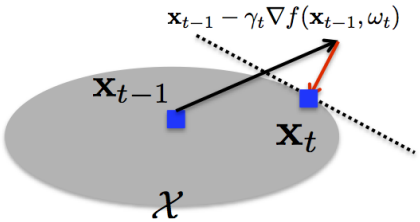
- Coverage:
  - SGD for various objectives
  - Accelerated SGD for Stochastic Composite Optimization
  - Variance Reduced SGD
  - Parallel and Distributed Optimization

Stochastic Gradient Descent

$$\min_{\mathbf{x}\in\mathcal{X}} f(\mathbf{x})$$

- stochastic gradient $\nabla f(\mathbf{x};\omega)$: ω is a random variable
- SGD updates:

$$x_t \leftarrow \Pi_{\mathcal{X}}\left[x_{t-1} - \gamma_t \partial f(x_{t-1};\omega_t)\right]$$



$$\Pi_{\mathcal{X}}[\hat{\mathbf{x}}] = \min_{\mathbf{x}\in\mathcal{X}} \|\mathbf{x} - \hat{\mathbf{x}}\|_2^2$$

Issues: How to determine learning rate γ_t? key: decreasing

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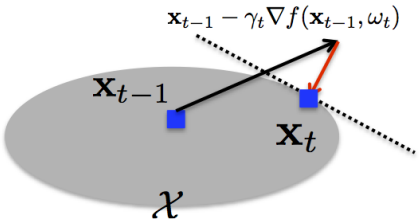

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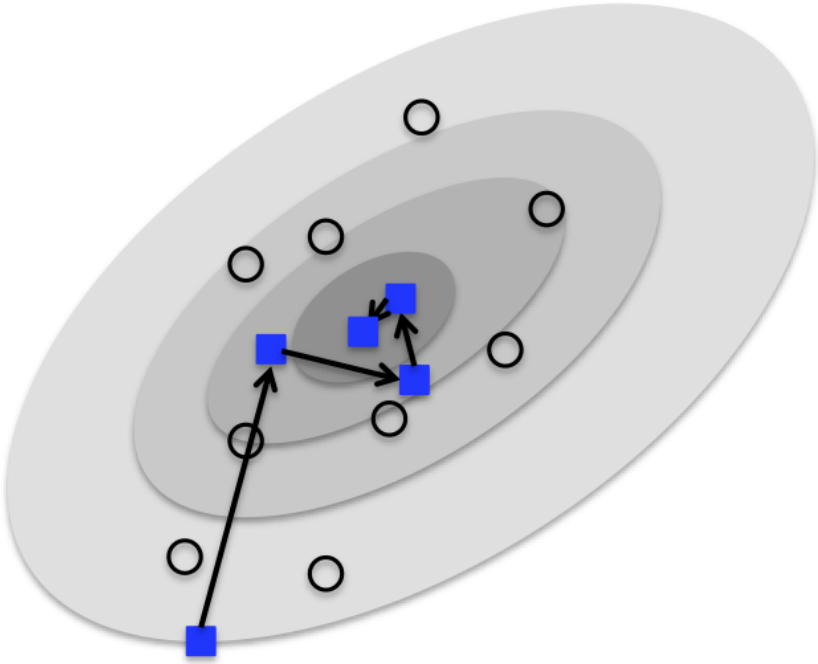
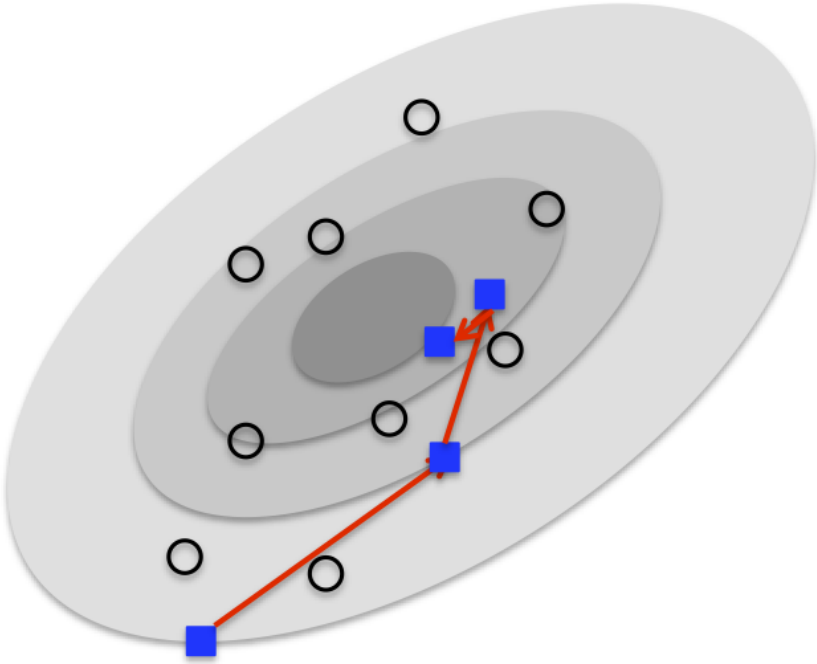
Why Stochastic Gradient Descent Works?

Why decreasing step size? GD vs SGD





$$\mathbf{x}_t = \mathbf{x}_{t-1} - \eta \nabla f(\mathbf{x}_{t-1})$$

$$\nabla f(\mathbf{x}_{t-1}) \to 0$$

$$\mathbf{x}_t = \mathbf{x}_{t-1} - \eta_t \nabla f(\mathbf{x}_{t-1}; \omega_t)$$

$$\nabla f(\mathbf{x}_{t-1}; \omega_t) \not\to 0, \quad \eta_t \to 0$$

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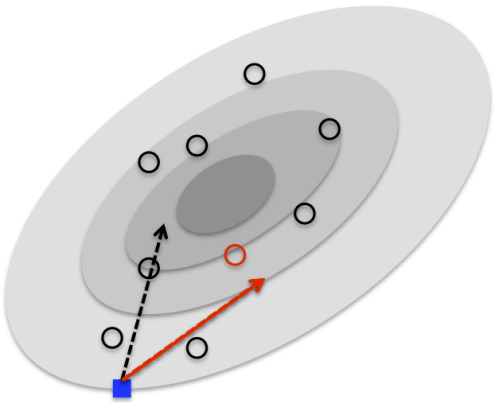

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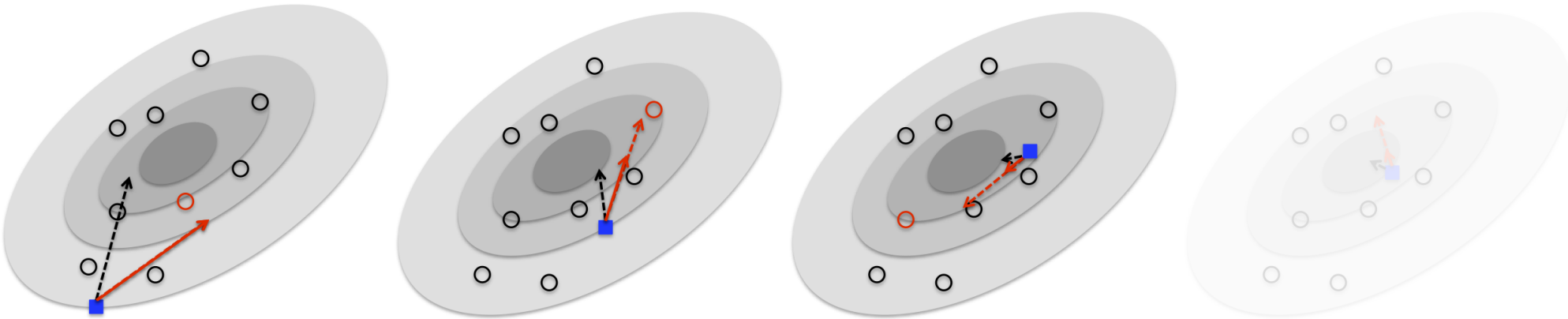

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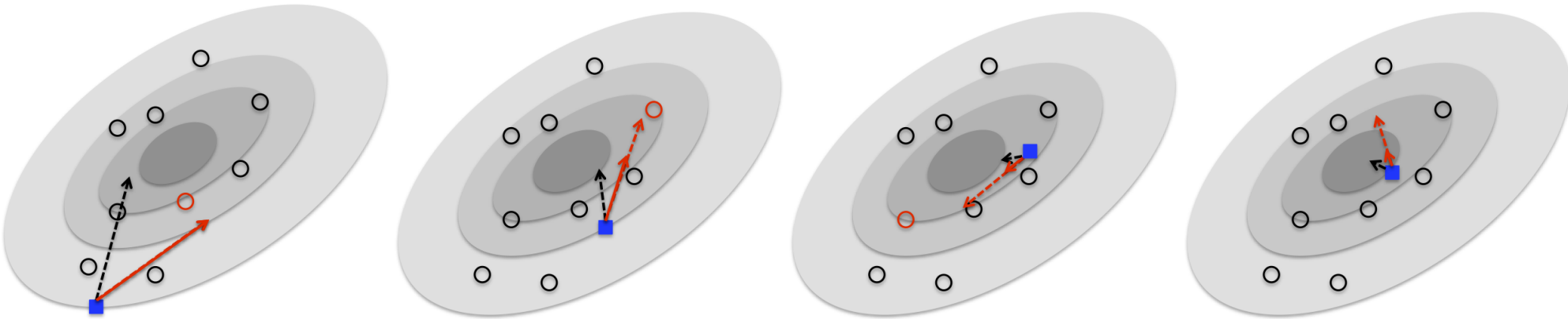

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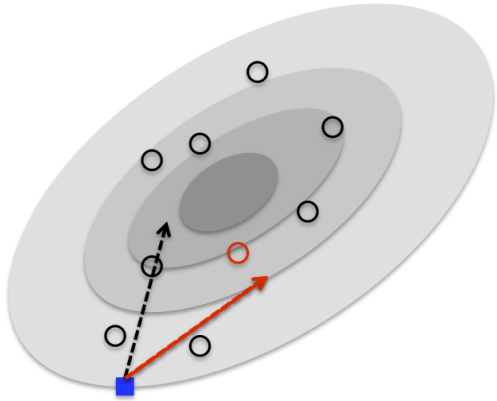

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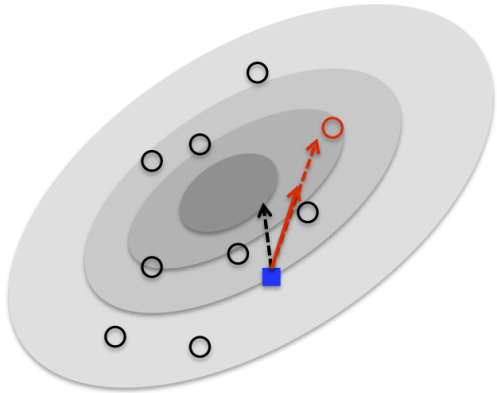

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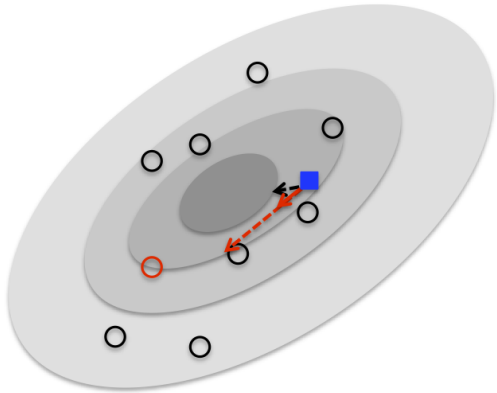

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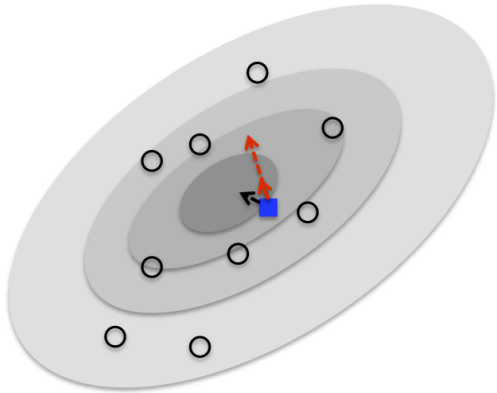

How to decrease the Step size?

Step size Depends on the property of the function.

SGD for Non-smooth Function: Convergence Rate

- $\|\mathbf{x} - \mathbf{y}\| \leq D$ and $\|\partial f(x;\omega)\|_* \leq G$, $\forall \mathbf{x}, \mathbf{y} \in \mathcal{X}$.
- Step size: $\gamma_t = \frac{c}{\sqrt{t}}$, c usually needed to be tuned
- Convergence rate of final solution $\mathbf{x}_T$: $O\left(\left(\frac{D^2}{c} + cG^2\right)\frac{\log T}{T}\right)$
- Are we good enough? Close but not yet
- minimax optimal rate : $O\left(\frac{1}{\sqrt{T}}\right)$

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SGD for Non-smooth Function: Convergence Rate

- minimax optimal rate : $O\left(\frac{1}{\sqrt{T}}\right)$
- Averaging:

$$\bar{\mathbf{x}}_t = \left(1 - \frac{1+\eta}{t+\eta}\right)\bar{\mathbf{x}}_{t-1} + \frac{1+\eta}{t+\eta}\mathbf{x}_t, \quad \eta \geq 0$$

- $\eta = 0$ standard average

$$\bar{\mathbf{x}}_T = \frac{\mathbf{x}_1 + \cdots + \mathbf{x}_T}{T}$$

- Convergence Rate of $\bar{\mathbf{x}}_T$: $O\left(\eta\left(\frac{D^2}{c} + cG^2\right)\frac{1}{T}\right)$

SGD for Non-smooth Strongly Convex Function: Convergence Rate

- λ-Strongly Convex minimax optimal rate : $O(\frac{1}{T})$
- Step size: $\gamma_t = \frac{1}{\lambda t}$
- Convergence Rate of $\mathbf{x}_T$: $O\left(\frac{G^2 \log T}{\lambda T}\right)$
- Averaging:

$$\bar{\mathbf{x}}_t = \left(1 - \frac{1+\eta}{t+\eta}\right)\bar{\mathbf{x}}_{t-1} + \frac{1+\eta}{t+\eta}\mathbf{x}_t, \quad \eta \geq 0$$

- Convergence Rate of $\bar{\mathbf{x}}_T$: $O\left(\frac{\eta G^2}{\lambda T}\right)$

SGD for Smooth function

- Generally No Further improvement upon $O\left(\frac{1}{\sqrt{T}}\right)$
- Special structure may yield better convergence
- Stochastic Composite Optimization: Smooth + Non-Smooth

General Strategies for Stochastic Optimization

General strategies for Stochastic Gradient Descent

- Coverage:
  - SGD for various objectives
  - Accelerated SGD for Stochastic Composite Optimization
  - Variance Reduced SGD
  - Parallel and Distributed Optimization

Stochastic Composite Optimization

$$\min_{\mathbf{x}\in\mathcal{X}} \underbrace{f(\mathbf{x})}_{\textit{smooth}} + \underbrace{g(\mathbf{x})}_{\textit{nonsmooth}}$$

- Example 1

$$\min_{\mathbf{w}} \frac{1}{n}\sum_{i=1}^{n} \log(1+\exp(-y_i\mathbf{w}^\top\mathbf{x}_i)) + \lambda\|\mathbf{w}\|_1$$

- Example 2

$$\min_{\mathbf{w}} \frac{1}{n}\sum_{i=1}^{n} \frac{1}{2}(y_i-\mathbf{w}^\top\mathbf{x}_i)^2 + \lambda\|\mathbf{w}\|_1$$

Stochastic Composite Optimization

- Accelerated Stochastic Gradient
  - auxiliary sequence of search solutions
  - maintain three solutions: $\mathbf{x}_t$ (action solution), $\bar{\mathbf{x}}_t$ (averaged solution), $\mathbf{x}_t^m$ (auxiliary solution)
  - Similar to Nesterov's deterministic method
  - several variants

- Stochastic Dual Coordinate Ascent

$$\min_{\mathbf{w}\in\mathbb{R}^d} \frac{1}{n}\sum_{i=1}^{n} \ell(\mathbf{w}^\top \mathbf{x}_i, y_i) + \sigma \sum_{g=1}^{K} \|\mathbf{w}_g\|_1 + \boxed{\frac{\lambda}{2}\|\mathbf{w}\|_2^2}$$

G. Lan's Accelerated Stochastic Approximation

Iterate $t = 1, \ldots, T$

1. compute auxiliary solution

$$\boxed{\mathbf{x}_t^m = \beta_t^{-1}\mathbf{x}_t + (1-\beta_t^{-1})\bar{\mathbf{x}}_t}$$

2. update solution

$$\mathbf{x}_{t+1} = P_{\mathbf{x}_t}(\gamma_t G(\mathbf{x}_t^m; \omega_t)) \text{ (think about projection)}$$

3. update averaged solution

$$\bar{\mathbf{x}}_{t+1} = (1-\beta_t^{-1})\bar{\mathbf{x}}_t + \beta_t^{-1}\mathbf{x}_{t+1} \text{ (Similar to SGD)}$$

γ_t, β_t, Convergence Rate?

Accelerated Stochastic Approximation

1. $\beta_t^{-1} = \frac{2}{t+1}$:

$$\bar{\mathbf{x}}_{t+1} = \left(1 - \frac{2}{t+1}\right)\bar{\mathbf{x}}_t + \frac{2}{t+1}\mathbf{x}_{t+1}$$

2. $\gamma_t = \frac{t+1}{2}\gamma$: Increasing, Never like before

3. Convergence rate of $\bar{\mathbf{x}}_T$:

$$O\left(\frac{L}{T^2} + \frac{M+\sigma^2}{\sqrt{T}}\right)$$

Variance Reduction: Mini-batch

- Batch gradient: $g_t = \frac{1}{b}\sum_{i=1}^{b} \partial f(x_{t-1}, \omega_{ti})$
- Unbiased sampling: $\mathbb{E}g_t = \mathbb{E}\partial f(x_{t-1}, \omega)$.
- Variance reduced: $\mathbb{V}g_t = \frac{1}{b}\mathbb{V}\partial f(x_{t-1}, \omega)$.
- Tradeoff the batch computation for variance decrease.
- Batch size
  1. Constant batch size;
  2. Increasing batch size as proportion to t.
- Convergence Rate: e.g. $O(\frac{1}{b\lambda T})$ of SGD for strongly convex

Variance Reduction: Mixed-Grad (Mahdavi et. al, Johnson et al. '13)

Iterate $s = 1, \ldots,$

Iterate $t = 1, \ldots, m$

$$\mathbf{x}_t^s = \mathbf{x}_{t-1}^s - \eta \underbrace{\left(\nabla f(\mathbf{x}_{t-1}^s; \omega_t) - \nabla f(\bar{\mathbf{x}}^{s-1}; \omega_t) + \nabla f(\bar{\mathbf{x}}^{s-1})\right)}_{StoGrad - StoGrad + Grad}$$

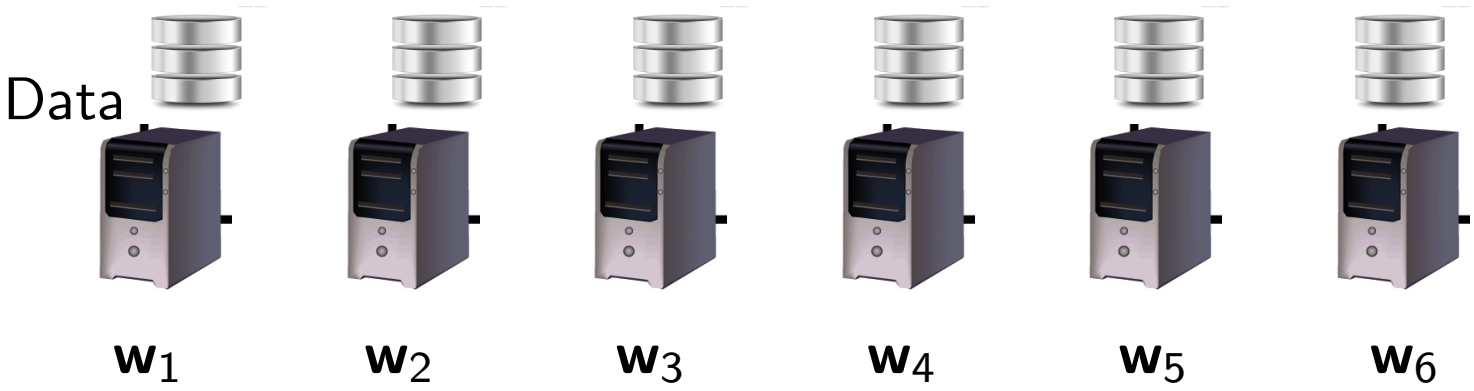
update $\bar{\mathbf{x}}^s$

- how variance is reduced?
  if $\bar{\mathbf{x}}^{s-1} \to \mathbf{x}^*$, $\nabla f(\bar{\mathbf{x}}_{t-1}) \to 0$, $\nabla f(\mathbf{x}_{t-1}; \omega_t) - \nabla f(\mathbf{x}^*; \omega_t) \to 0$
- $\bar{\mathbf{x}}^s = \mathbf{x}_m^s$ or $\bar{\mathbf{x}}^s = \sum_{t=1}^m \mathbf{x}_t^s / m$
- Constant learning rate
- Better Iteration Complexity.: $O((n + \frac{1}{\lambda}) \log\left(\frac{1}{\epsilon}\right))$ for sm&sc, $O(1/\epsilon)$ for smooth

Distributed Optimization: Why?

- data distributed over multiple machines
- moving to single machine suffers
  - low network bandwidth
  - limited disk or memory
- benefits from parallel computation
  - cluster of machines
  - GPUs

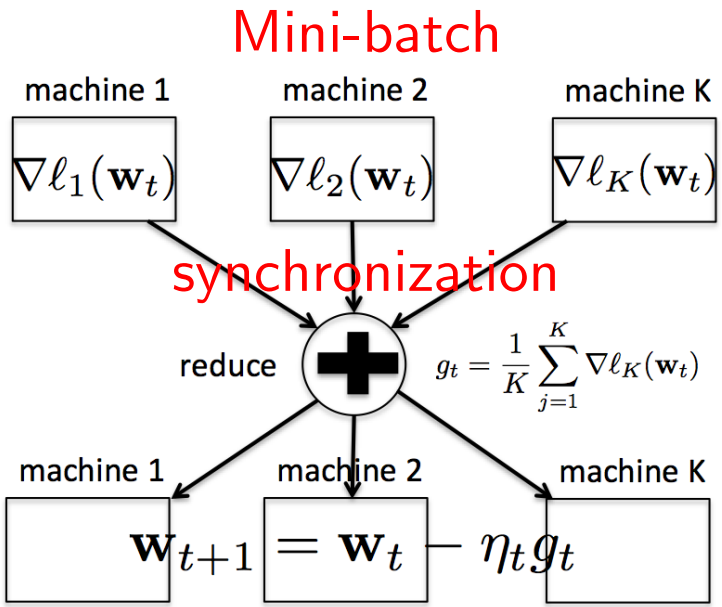
A Naive Solution



$$\mathbf{w} = \frac{1}{k} \sum_{i=1}^{k} \mathbf{w}_i$$

Issue: Not the Optimal

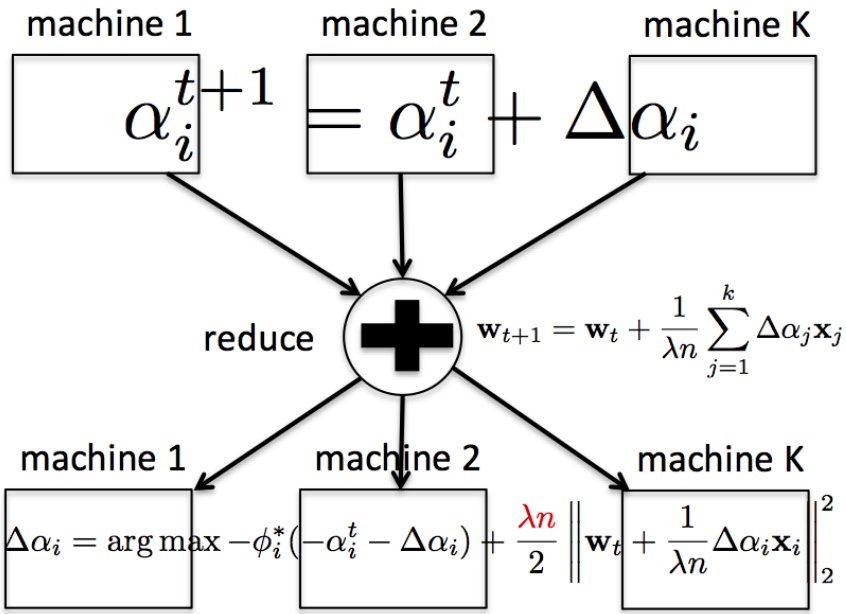
Distribute SGD is simple



Mini-batch SGD

- Good: reduced variance, faster conv.
- Bad: synchronization is expensive
- Solutions:
  - asynchronized update
    - convergence: difficult to prove
  - lesser synchronizations
    - guaranteed convergence

Distribute SDCA is not trivial



issue: data are correlated

$$\Delta\alpha_i = \arg\max -\phi_i^*(-\alpha_i^t - \Delta\alpha_i) - \frac{\lambda n}{2}\left\|\mathbf{w}_t + \frac{1}{\lambda n}\Delta\alpha_i\mathbf{x}_i\right\|_2^2$$

Not Working

Distribute SDCA (T. Yang NIPS'13)

The Basic Variant

mR-SDCA running on machine k

Iterate: for $t = 1, \ldots, T$

> **Iterate:** for $j = 1, \ldots, m$
>
> Randomly pick $i \in \{1, \cdots, n_k\}$ and let $i_j = i$
>
> Find $\Delta\alpha_{k,i}$ by **IncDual**($\mathbf{w} = \mathbf{w}_{t-1}$, $scl = mK$)
>
> Set $\alpha_{k,i}^t = \alpha_{k,i}^{t-1} + \Delta\alpha_{k,i}$

Reduce: $\mathbf{w}_t \leftarrow \mathbf{w}_{t-1} + \frac{1}{\lambda n}\sum_{j=1}^m \Delta\alpha_{k,i_j} x_{k,i_j}$

$$\Delta\alpha_i = \arg\max -\phi_i^*(-\alpha_i^t - \Delta\alpha_i) - \frac{\lambda n}{2mK}\left\|\mathbf{w}_t + \frac{mK}{\lambda n}\Delta\alpha_i \mathbf{x}_i\right\|_2^2$$

Distribute SDCA (T. Yang NIPS'13)

The Practical Variant

Initialize: $\mathbf{u}_t^0 = \mathbf{w}^{t-1}$

Iterate: for $j = 1, \ldots, m$

Randomly pick $i \in \{1, \cdots, n_k\}$ and let $i_j = i$

Find $\Delta\alpha_{k,i_j}$ by **IncDual**($\mathbf{w} = \mathbf{u}_t^{j-1}$, $scl = k$)

Update $\alpha_{k,i_j}^t = \alpha_{k,i_j}^{t-1} + \Delta\alpha_{k,i_j}$

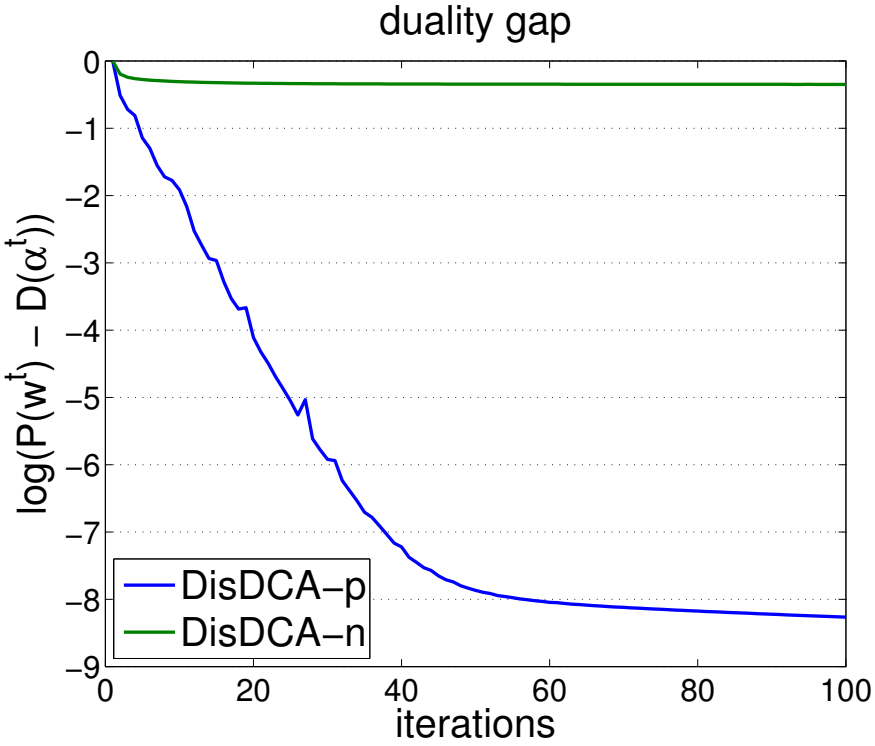
Update $\mathbf{u}_t^j = \mathbf{u}_t^{j-1} + \frac{1}{\lambda n_k}\Delta\alpha_{k,i_j}\mathbf{x}_{k,i_j}$

$$\Delta\alpha_{i_j} = \arg\max -\phi_{i_j}^*(-\alpha_{i_j}^t - \Delta\alpha_{i_j}) - \frac{\lambda n}{2K}\left\|\mathbf{u}_t^{j-1} + \frac{K}{\lambda n}\Delta\alpha_{i_j}\mathbf{x}_{i_j}\right\|_2^2$$

Using Updated Information and Large Step Size

Distribute SDCA (T. Yang NIPS'13)

The Practical Variant is much faster than the Basic Variant



Outline

Implementations and A Library

In this section, we present some techniques for efficient implementations and a practical library

- Coverage:
  - efficient averaging
  - Gradient sparsification
  - pre-optimization: screening
  - pre-optimization: data preconditioning
  - distributed (parallel) optimization library

Efficient Averaging

- Update rule:

$$x_t = (1 - \gamma_t \lambda) x_{t-1} + \gamma_t g_t$$
$$\bar{x}_t = (1 - \alpha_t) \bar{x}_{t-1} + \alpha_t x_t$$

- Efficient update when g_t has many 0, or g_t is sparse,

$$S_t = \begin{pmatrix} 1 - \lambda\gamma_t & 0 \\ \alpha_t(1 - \lambda\gamma_t) & 1 - \alpha_t \end{pmatrix} S_{t-1}, \quad S_1 = I$$
$$y_t = y_{t-1} - [S_t^{-1}]_{11} \gamma_t g_t$$
$$\tilde{y}_t = \tilde{y}_{t-1} - ([S_t^{-1}]_{21} + [S_t^{-1}]_{22} \alpha_t) \gamma_t g_t$$

$$x_T = [S_T]_{11} y_T$$
$$\bar{x}_T = [S_T]_{21} y_T + [S_T]_{22} \tilde{y}_T$$

When Gradient is Sparse

Gradient sparsification

- Sparsification by importance sampling

$$R_{ti} = \mathrm{unif}(0, 1)$$
$$\tilde{g}_{ti} = g_{ti}[|g_{ti}| \geq \hat{g}_i] + \hat{g}\mathrm{sign}(g_{ti})[\hat{g}_i R_{ti} \leq |g_{ti}| < \hat{g}_i]$$

- Unbiased sample: $\mathbb{E}\tilde{g}_t = g_t$.
- Tradeoff variance increase for the efficient computation.

Especially useful for Logistic Regression

Pre-optimization: Screening

- Screening for Lasso (Wang et al., 2012)
- Screening for SVM (Ogawa et al., 2013)

Distributed Optimization Library: Birds

- The birds library implements distributed stochastic dual coordinate ascent (DisDCA) for classification and regression with a broad support.
- For technical details see:
  - "Trading Computation for Communication: Distributed Stochastic Dual Coordinate Ascent." Tianbao Yang. NIPS 2013.
  - "On the Theoretical Analysis of Distributed Stochastic Dual Coordinate Ascent" Tianbao Yang, etc. Tech Report 2013.
- The code is distributed under GNU General Publich License (see license.txt for details).

Distributed Optimization Library: Birds

What problems does it Solve?

- Classification and Regression
- Loss
  1. Hinge loss (SVM): L-1 hinge loss, L-2 hinge loss
  2. Logistic loss (Logistic Regression)
  3. Least Square Regression (Ridge Regression)
- Regularizer
  1. ℓ_2 norm: SVM, Logistic Regression, Ridge Regression
  2. ℓ_1 norm: Lasso, SVM, LR with ℓ_1 norm
- Multi-class : one-vs-all

Distributed Optimization Library: Birds

What data does it Support?

- dense, sparse
- txt, binary

What environment does it Support?

- Prerequisites: Boost Library
- Tested on A cluster of Linux Servers (up to hundreds of machines)
- Tested on Cygwin in Windows with multi-core

How about kernel methods?

- Stochastic/Online Optimization Approaches: (Jin et al., 2010; Orabona et al., 2012),

- Linearization + STOP for linear methods
  - the Nyström method (Drineas & Mahoney, 2005)
  - Random Fourier Features (Rahimi & Recht, 2007)
  - Comparison of two (Yang et al., 2012)

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