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A Behind-the-Scenes View of the Development of Algebra

Negative Math: How Mathematical Rules Can Be Positively Bent. *By Alberto A. Martinez, Princeton University Press, Princeton, New Jersey, 2006, xii + 267 pages, $24.95.*

BOOK REVIEW

By James Case

Alberto Martinez—a professor of the history of science and mathematics at the University of Texas—has written an entire book about the fact that the product of two negative numbers is considered positive. He begins by reminding his readers that it need not be so. Only after extended debate and genuine soul-searching did the early (European) algebraists settle on that convention. Girolamo Cardano, among others, objected. Why, he asked, should the introduction of negative numbers cause positive numbers as familiar as 4, 9, and 16 to acquire a second square root? Did they really need more than one? Euler, in his *Complete Introduction to Algebra* (1770), explained positive and negative numbers in monetary terms, arguing that because owing m dollars to n creditors was equivalent to owing mn dollars to one, $-m \times n$ should equal $-(m \times n)$. But he cited no similarly compelling reason* for which $-m \times -n$ should equal $m \times n$. To show how the subject might have developed along the lines suggested by Cardano, Martinez develops an apparently viable arithmetic (seemingly free of any internal contradiction) in which the alternative convention prevails.

Until the middle of the 19th century, many prominent figures continued to deny the very possibility of "quantities less than zero." Although the practice of distinguishing debits from credits by enclosing the former in parentheses—or by recording them in red ink—was already well established among bookkeepers and accountants, "natural philosophers" tended to doubt that their use could be logically justified. Is it possible, Pascal asked, to extract seven stones from a hat containing only three? Can one who concedes that -4 is a legitimate number then deny that its square roots are equally legitimate? As late as the mid-19th century, those convinced that mathematics is and should ever remain a "science of measurable quantities" continued to refer to numbers like -4 and its square root as "impossible numbers" and "obvious absurdities." Some of the most troubling issues lay in the realm of ratio and proportion—so important a part of Euclid's geometry. It was argued, for example, that the equation –1:1 = 1:–1 is as absurd as 2:4 = 4:2, because each asserts that the ratio of the smaller of two numbers to the larger equals that of the larger to the smaller.

The debates surrounding the arithmetic of negative, imaginary, and even infinitesimal numbers overlapped to a considerable extent. The positive reals—but none of the others—can all be realized as distances between distinct points in the Euclidean plane, as elapsed times, as the masses of "ponderable" bodies, and so on. Martinez strives to convey the genuine angst felt by the leading participants in these disputes, in part by quoting them frequently and at length. Fermat, Descartes, Wallis, Maclaurin, Leibniz, Johann Bernoulli, d'Alembert, Kant, Bishop Berkeley (one of Newton's most coherent critics), and no small contingent of others are all allowed to speak their piece. Difficult though it is to imagine a subject less likely to hold an experienced reader's interest than an extended discussion of arithmetic signs, Martinez does well in this regard.

When it came, the acceptance of negative, imaginary, and complex numbers was both swift and complete. It was triggered mainly, according to Martinez, by William R. Hamilton's development of quaternions. As soon as they recognized that complex numbers correspond to points of the Euclidean plane in a way that permits them to be combined algebraically, mathematicians began to regret their inability to combine the points of Euclidean 3-space in like manner. Hamilton's quaternions were an attempt to do just that. Yet many of his contemporaries found Hamilton's solution to the problem unsatisfactory in several respects. They objected to the fact that the multiplication of quaternions need not be commutative, and found it offensive that the squares of quaternions are frequently negative. Vector algebra, as subsequently formulated by J.W. Gibbs in the U.S. and O. Heaviside in the U.K., seemed a more natural algebra of points in 3-space. Nevertheless, it was Hamilton's quaternions that—more than anything else—opened

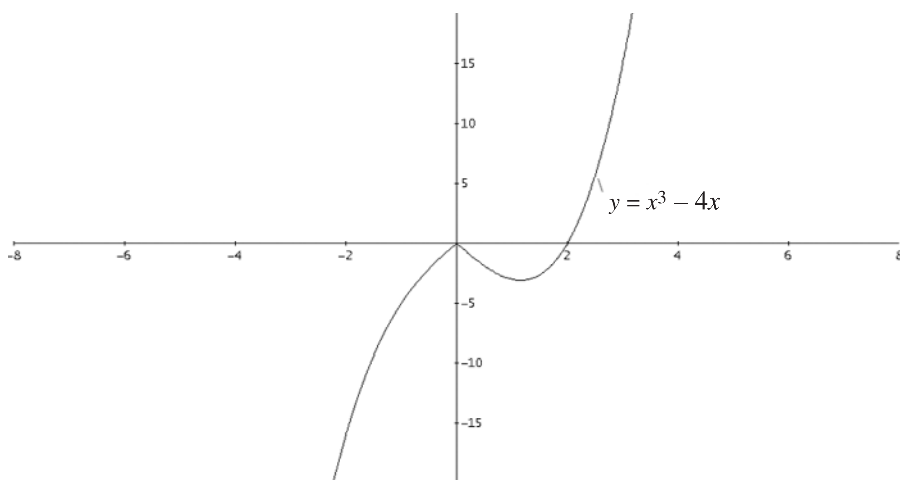

Figure 1. *Simple functions take unfamiliar forms in an alternative algebra explored in the book under review.*

*The reviewer is particularly sympathetic to this issue, having once attempted to explain negative numbers to a class of eighth graders. They bought Euler's argument that $-m \times n$ should equal $-(m \times n)$, but refused to be convinced that $-m \times -n$ should equal $m \times n$.

the eyes of the mathematical community to a new and more abstract way of studying algebra.

The algebra explored by Martinez, in which the product of two negative numbers is again negative, offers certain advantages. In it, x, x^2, x^3, . . . are all of the same sign, for both positive and negative values of x. Hence, every number of either sign has a unique real root of every (positive integral) order. Likewise, positive numbers have unique positive logarithms to any base b > 1, while negative numbers have unique negative logarithms. Moreover, because high powers of x still dominate lower ones, every monic polynomial possesses at least one real root. On the other hand, $ax^2 - 1$ is negative for all values of x when $a = -2$. Martinez explores the resulting algebra in some detail, finding it surprisingly handy in some ways, though awkward in others. For one thing, the graphs of simple functions assume unfamiliar forms, as shown in Figure 1.

Martinez also explores the convention whereby the product of two real numbers inherits the sign of the first factor, regardless of the sign of the second. In the resulting algebra, $x^3 + 2x^2 - 8x$, $x^3 + 2x^2 - x8$, $x^3 + x^22 - 8x$, and $x^3 + x^22 - x8$ represent four different polynomials with four different graphs. Without giving any indication that the likes of Cardano and Euler ever considered such an option—it seems unlikely, given the resistance to non-commutative multiplication encountered by Hamilton, that they gave it a moment's thought—Martinez rightly points out that there is no strictly logical reason they could not have done so. Indeed, that may be the real "take-home" message of his entire book: Logic alone dictates surprisingly little of what gets done. Intuition, experience, and apparent convenience do more than we realize to shape mathematical thoughts and deeds.

The book is written in a relaxed, conversational manner calculated to make it accessible to any high school student of algebra, at the expense of frustrating the more experienced reader. The latter will wish for a more compact presentation. It can be recommended to anyone with an interest in the way algebra developed behind the scenes, at a time when calculus and analytic geometry were the main focus of mathematical interest.

James Case writes from Baltimore, Maryland.