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There's More Than One Way to Scan a CAT . . .

By Barry A. Cipra

I think I see the light coming to me, coming through me giving me a second sight.—Cat Stevens

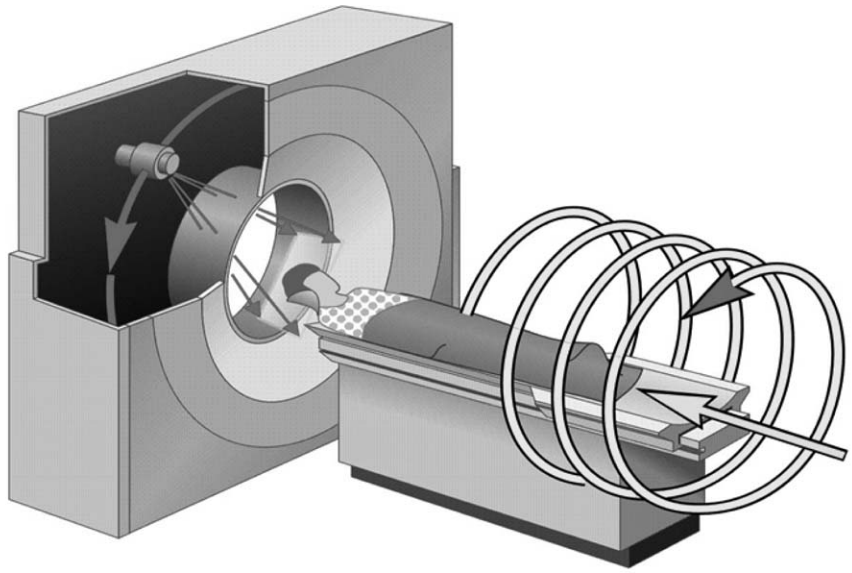

Figure 1. *Rotation + Translation = Helix. When a patient on a table slides through a CT machine with a constantly circling x-ray source, the net effect is that of an x-ray machine spiraling the length of the patient.*

CAT scans, introduced in the 1970s, have saved or improved the lives of innumerable people. A staple of modern medical science, they are also widely used in industry to check the integrity of hard-to-get-at pipes and processes. In many ways, computational tomography is a "mature" technology.

In some respects, however, computerized tomography remains at the (non-invasive) cutting edge of research. In particular, the mathematical understanding of tomography continues to improve. Alexander Katsevich of the University of Central Florida, an invited speaker at the 2006 SIAM Conference on Imaging Science, has been a leader in the development of new theoretical tools and practical algorithms for obtaining high-quality images from the attenuation of high-energy photons doing their damnedest to travel along straight lines.

The original CAT-scan machines were clunky affairs that made the most of two technological limitations. One limitation was that x-ray sources at the time could work only in short bursts. The other was that mathematicians had thoroughly worked out the practicalities of inverting the Radon transform only in two dimensions. The upshot was axial tomography: The machine would painstakingly take individual images from different angles in a single plane, which the inverse Radon transform would turn into a picture showing the planar "tomos" (Greek for "slice") through the patient. For a complete picture of a patient's innards, the patient would be slowly moved through the machine, a few millimeters at a time, along an axis perpendicular to the slices. In short, computed axial tomography makes images in much the same way a deli owner shaves ham for sandwiches.

The problem was, it took a long time to do a scan. Part of the down time arose from the need for smooth ac- and de-celeration of the patient at each repositioning—patients would not find it pleasant to be repeatedly jerked. All this changed in the 1990s, when researchers developed x-ray machines capable of scanning continuously. It was now possible to put the x-ray source and detectors across from each other on a spinning ring and slide the patient through at a comfortable, constant rate (see Figure 1). The new alternative is known as helical CT: From the patient's point of view, the x-ray/detector apparatus traces out a helix from head to toe. (The technical jargon has dropped the "A" from "CAT" because slices are no longer necessarily taken along an axis, but the original name has persisted.)

Helical (also called spiral) CT has been a boon to patients and doctors, but a challenge to mathematicians. The relatively simple mathematics of the two-dimensional Radon transform is no longer sufficient, except as a rough approximation. (The smaller the pitch of the helix, the better the approximation.) At the same time, engineers were developing more sophisticated CT machines with larger pitch angles. It was incumbent on mathematicians to rethink the inverse problem.

Katsevich has done much of the rethinking. In 2001, he found an exact solution for the helical CT problem. Two technological developments made his solution practical: widening the detector array from a single strip of photon detectors to a rectangle and masking the x-ray source so that instead of simply fanning out it spreads in a cone. With a conical beam, every point within the patient receives radiation along lines in a continuum of directions as the patient and x-ray source both move. The helical path of the combined motion makes the inverse problem fully three-dimensional, but the regularity of the trajectory renders the mathematical analysis tractable.

Theoretically exact solutions are sometimes worthless in practice. But Katsevich's solution lends itself to efficient algorithmic implementation, and it copes with issues of noise and finite resolution. (Earlier algorithms, created by researchers whose analyses paved the way for Katsevich's discovery, were either exact but impractical or practical but approximate.) Moreover, it reduces to the usual 2D Radon transform as the helical pitch tends to zero, or when the object being scanned is constant along the spiral's axis.

Katsevich's algorithm involves a filtering step, followed by back projection. The underlying mathematics is partly geometric and partly analytic. The geometric portion hinges on the fact that every point P inside the cylindrical core of the helix—in particular, every point P inside the patient—lies on precisely one "PI line." A PI line connects two points of the helix that are within one turn. "PI" in the name connotes the number π: From the standpoint of *P*, the x-ray source rotates through exactly 180 degrees (i.e., π radians) as it traverses the helix from one endpoint of *P*'s PI line to the other. This portion of the helix is referred to as *P*'s "helical π-segment."

The analytic portion boils down to showing that the reconstruction at *P*—that is, solving for the amount contributed by whatever's at *P* to the attenuation of the x-ray—depends only on data taken from carefully selected planes through *P* as the source traverses *P*'s helical π-segment. A further geometric portion of the algorithm determines the minimum size of the detector—in particular, the minimum width—needed to ensure that all rays within the important planes are detected. This minimal region is obtained using the Tam–Danielsson window, named for the researchers who independently discovered the significance of different aspects of PI lines in the mid-1990s: Kwok C. Tam, then at the General Electric Company in Schenectady, New York, and Per-Erik Danielsson of Linköping University in Sweden.

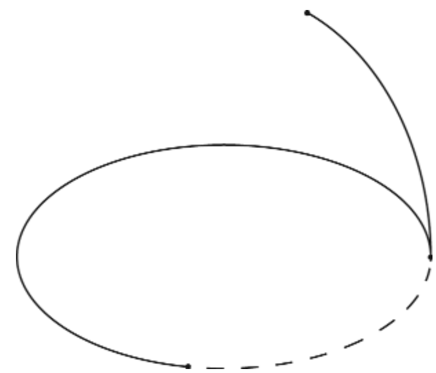

Figure 2. *Circle-and-arc trajectory. The latest in CT technology has the machine do all the moving. A common trajectory combines most or all of a circle with an arc at right angles to the circle. Courtesy of Alexander Katsevich.*

With larger detectors (or an x-ray source that rotates more quickly), the Tam–Danielsson computation implies that the patient can be slid through the CT machine more quickly. This is a good thing, up to a point: There are limits to how fast a patient can be comfortably and safely scanned. Katsevich has developed variant algorithms that utilize redundant data produced with the pitch of the helix about three times tighter than required for the Tam–Danielsson window. The factor of three opens up a second parametric interval that, blended with the first, makes the reconstruction less susceptible to sampling artifacts and noise.

What about other trajectories? One that's used with certain newer, portable CT machines is known as the circle-and-arc trajectory: The apex of the conical beam (i.e., the x-ray source) traces a circle and then part of a second circle, perpendicular to the first (see Figure 2). These machines, called C-arms (because of their shape), are used in surgery and other settings in which the patient and table can't be moved.

Katsevich has solved the corresponding inverse problem for this motion as well. The analysis again depends on properties of PI lines passing through points to be reconstructed, and the methods are general enough to work when the first circle is incomplete (meaning that the machine doesn't need to go all the way around) and the second arc lies along an ellipse—or even a straight line—rather thana circle.

The helical and circle-and-arc solutions raise a pair of questions, Katsevich says. First, which conical beam trajectories lend themselves to exact (and practical) solution? Second, is there a general framework that incorporates all solvable trajectories? The formulas for the helical case, for example, vary smoothly within a range of pitch and cone angles, but they are substantially different from the formulas for the circle-and-arc trajectories, portions of which Katsevich had to derive entirely anew. If medical imagers want to experiment with more trajectories—especially if they want to look for optimal CT techniques—it would be nice not to have to start from scratch each time. Just as patients benefit from the insights of CAT scan images, computational tomography continues to benefit from the insights of applied mathematics.

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