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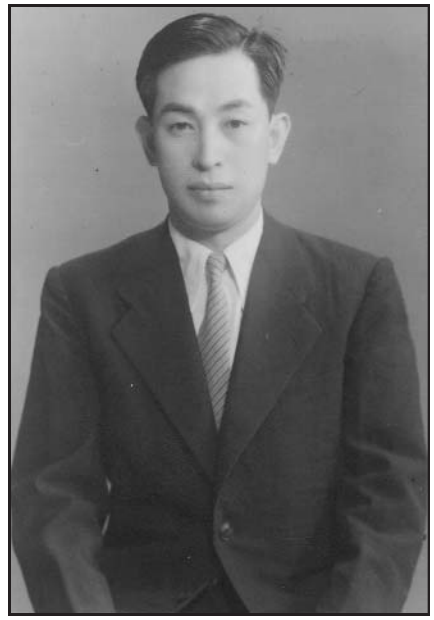
First Gauss Prize Is Awarded to Kiyoshi Itō in Madrid

By James Case



In 1954, at the age of 39, Kiyoshi Itō gave a series of lectures at the Institute for Advanced Study in Princeton. At the time, the stochastic calculus he had developed was already known in the West.

To improve public awareness of mathematics, and of the ways in which it enriches science, technology, and—indirectly—daily life, the Deutsche Mathematiker–Vereinigung, together with the International Mathematical Union, has created a new prize. Administered by the DMV and named for Carl Friedrich Gauss, the prize consists of a medal and a cash prize, currently valued at €10,000. The prize was awarded for the first time this year, at the opening ceremony of the International Congress of Mathematicians in Madrid. The endowment for the prize comes from a surplus generated by the 1998 ICM (Berlin).

Kiyoshi Itō was selected by the DMV to be the first recipient of the new prize. Born in Hokusei (now Inabe), Japan, on September 7, 1915, Itō graduated from the Imperial University at Tokyo in 1939. He accepted a position at the national statistical office, where he was encouraged to continue his studies toward the PhD (which he earned in 1945) and to publish his findings. Seven years later, he was named professor at the University of Kyoto, where he remained until his retirement in 1979. The prize was presented by Juan Carlos, King of Spain, to Itō's youngest daughter, Junko Itō, a professor of phonology at the University of California, Santa Cruz, who accepted on her father's behalf.

The disruptions of World War II—during which Imperial Japan endured severe shortages of almost everything, including paper to write on—delayed recognition of Itō's early achievements. Yet by 1954, in the course of which he gave a series of invited lectures at the Institute for Advanced Study in Princeton, his deeds were known in the West.

In particular, if $X(t)$ represents the perpendicular distance between a marked particle undergoing Brownian motion and a plane fixed in space, $X(t)$ will arguably vary in such a way that (i) the distribution of the random increments $X(b) - X(a)$ will depend only on the duration of the time interval $a \leq t \leq b$; (ii) the mean of that distribution will be zero; (iii) the variance of the distribution will be an increasing function of $b - a$; and (iv) successive increments $X(b) - X(a)$, $X(c) - X(b)$, $X(d) - X(c)$, . . . will be *statistically independent* of one another. If $h = b - a$ and $k = c - b$, the variance in question should thus satisfy the equation Var $(h + k)$ = Var (h) + Var (k). Hence, Var (h) must be of the form $\sigma^2 h$, for some positive constant σ. Itō developed a calculus applicable to all such processes $X(t)$, the most important of which are the Wiener processes, for which all increments are drawn from Gaussian normal distributions.

Itō's calculus makes it possible to integrate and, in a sense, differentiate such processes. Integrals of the form $\int_a^b f(X(t))dX(t)$ do not exist as ordinary Stieltjes integrals, because $X(t)$ is almost surely of unbounded variation over every interval $a \leq t \leq b$. A value can be ascribed to such integrals in any of several different ways, however. Doing it Itō's way leads to the conclusion that

$$\int_a^b X(t)dX(t) = ½\,(X^2(b) - X^2(a)) - ½(b - a), \tag{1}$$

the extra term being due to the unusual nature of the integrator. Although other definitions have been proposed—and do have their uses—Itō's stochastic calculus seems to be the best known.

During the 1930s, Wiener and Kolmo-goroff had developed a theory of filtering and prediction for objects whose equations of motion were of the form $ds(t)/dt = f(s(t), x(t))$, where $x(t)$ was an irregular "disturbance term." Apparently aware of their efforts, Itō asked in effect what would happen if $x(t)$ were replaced by a stochastic process $X(t)$ of the sort described above. His ability to integrate such processes made it possible to analyze the integrated forms of the simplest of the foregoing "stochastic differential equations," including the one in which $f(s,x) = s(\alpha + \beta x)$, for constants α and β. The result is

$$S(t) = S(t_o) + \alpha \int^t S\,dt + \beta \int^t S\,dX, \tag{2}$$

where $s(t)$ has been replaced by $S(t)$ in rec-ognition that it too must now be stochastic. In fact, it is only the integrated forms of such equations that have any clear meaning, because stochastic processes of the sort described above almost never have derivatives. Itō compromised by writing (2) in the form

$$dS = \alpha S\,dt + \beta S\,dX. \tag{3}$$

It too is meaningless until integrated, but at least avoids the notations dS/dt and dX/dt, which many would find offensive in the present context. When $\beta = 0$, (3) reduces to the familiar $dS/dt = \alpha S$, which implies exponential growth or decay at the constant rate α.

By far the most celebrated application of the Itō calculus is to the evaluation of stock options by means of the Black–Scholes equation. This is done by assuming that the price $S(t)$ of a share of stock in XYZ corporation satisfies a stochastic differential equation of the form (3) in which $\alpha = r + \mu$, where r is a risk-free rate of interest and μ represents the "risk premium" a risky asset must offer in order to lure investors away from the risk-free alternative, and $X = X(t)$ denotes a Wiener process with unit variance. In such circumstances, β becomes a measure of the riskiness of the stock in question. Financial markets impute a definite value V to an option permitting the holder to buy a share of the stock in question at an exercise price K on a specified date T. That value is observed to vary with the price $S(t)$ of the "underlying" stock, as well as with time. Assuming V to depend on no other variables, it must satisfy a related stochastic differential equation of the form

$$dV(t, S(t))/dt = V_t(t,S(t)) + V_S(t,S(t))dS(t)/dt. \tag{4}$$

It follows from (4) that the worth W of the portfolio formed by buying a single share of XYZ corporation and selling (short if necessary) N shares of the underlying stock satisfies

$$dW(t,S(t))/dt = V_t(t,S(t)) + [V_S(t,S(t)) - N]dS(t)/dt. \tag{5}$$

The key point is that (5) ceases instantaneously to be stochastic when $N = V_S(t,S(t))$ because the coefficient of $dS(t)/dt$ vanishes at such instants.

If it is now assumed that the "law of one price" holds instantaneously, so that portfolios that are instantaneously risk-free must earn an instantaneous rate of return equal to the prevailing rate of return on permanently risk-free assets, then the instantaneous rate $1/W(t,S(t)) \times dW(t,S(t))/dt$ must reduce in-stantaneously to r, the riskless rate of return, whenever $N = V_S(t,S(t))$. At such instants, $rW(t,S(t)) = r[V(t,S(t)) - V_S(t,S(t))S(t)]$, causing $V(t,S)$ to satisfy

$$V_t + rSV_S = rV$$
and
$$V(T,S) = \max(S - K,0) \tag{6}$$

at every point (t,S) through which a solution of (3) can pass. Accordingly, the unknown function $V = V(t,S)$ can be determined by solving a simple boundary value problem.

The foregoing program has never been successfully carried out, in large part because the quantities $dS(t)/dt$ that appear in (4) and (5) do not really exist. The Itō calculus, however, can be used to obtain an alternative form of (4) that leads to an alternative—yet equally soluble—boundary value problem (6) for the unknown function $V(t,S)$. The result is known as the Black–Scholes formula for the value of a European call option. American call options differ from their European cousins only in that the former can be exercised at any time $t \leq T$, whereas the latter can be exercised only on the actual expiration date. The Black–Scholes formula opened the door to a new age of financial risk management, and it is hard to imagine how all that could have happened without Itō's calculus.

James Case writes from Baltimore, Maryland.