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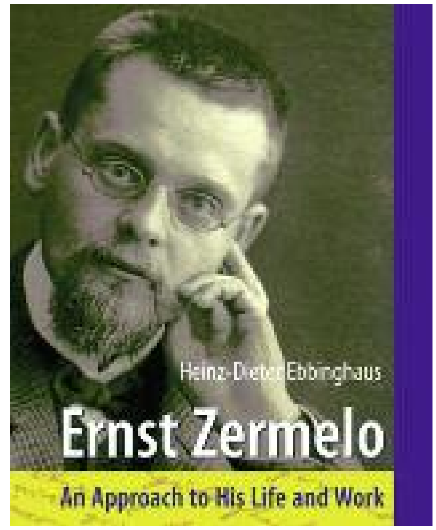
The Z of ZF and ZFC and ZF¬C

Ernst Zermelo: An Approach to His Life and Work. *By Heinz-Dieter Ebbinghaus, Springer, New York, 2007, 356 pages, $64.95.*

BOOK REVIEW

By Philip J. Davis



The Z, of course, is Ernst Zermelo (1871–1953). The F, in case you had forgotten, is Abraham Fraenkel (1891–1965), and the C is the notorious Axiom of Choice. The book under review, by a prominent logician at the University of Freiburg, is a splendid in-depth biography of a complex man, a treatment that shies away neither from personal details nor from the mathematical details of Zermelo's creations.

Zermelo was a Berliner whose early scientific work was in applied mathematics: His PhD thesis (1894) was in the calculus of variations. He had thought about this topic for at least ten additional years when, in 1904, in collaboration with Hans Hahn, he wrote an article on the subject for the famous *Enzyclopedia der Mathematische Wissenschaften*. In point of fact, though he is remembered today primarily for his axiomatization of set theory, he never really gave up applications.

In his Habilitation thesis (1899), Zermelo was arguing with Ludwig Boltzmann about the foundations of the kinetic theory of heat. Around 1929, he was working on the problem of the path of an airplane going in minimal time from A to B at constant speed but against a distribution of winds. This was a problem that engaged Levi-Civita, von Mises, Carathéodory, Philipp Frank, and even more recent investigators. The year 1932 found Zermelo working on an electrodynamic clutch for cars. Zermelo's scientific quiver thus held two arrows—applied mathematics and set theory—and they seemed to have little to do with each other in his mind or, for that matter, in the minds of applied mathematicians.

At the turn of the 20th century, Cantorian set theory was very much in the mathematical air. Hilbert was interested in foundational questions, and both Cantor and Hilbert were attracted to the question of the well-ordering of the reals. Is it possible to create a reordering of the real numbers in such a way that every subset has a first element? Zermelo put his mind to it and answered that yes, he could do it, granted the legitimate existence of what is now called a "choice function." Thus in 1904 was born the Axiom of Choice, and thus also, in the same year, as discussed shortly, was opened a philosophical Pandora's box, either of logical possibilities or ambiguities, or logical monstrosities—take your pick. Thus also was born Zermelo's axiomatization of set theory, now known as Z. Ebbinghaus lists the eight axioms both in the original German and in an English translation.

In the heady years that followed (Einstein's theory of special relativity dates to 1905), Zermelo, because of very poor health, was in and out of various professorships and sanitaria. Ultimately, in 1926, he was awarded an honorary university professorship at the University of Freiburg, by way of recognition and as a stopgap. In those years, Ebbinghaus suggests, and with good reason, that even as he proceeded with the development of infinitary logic, Zermelo anticipated Russell's paradox, game theory, and von Neumann's treatment of the ordinal numbers.

In 1921, Abraham Fraenkel, who started his mathematical life with research on *p*-adic numbers and ring theory, joined in the axiomatic game. He suggested that one of Zermelo's axioms in Z ought to be augmented. Zermelo agreed; he added axioms of "replacement" and of "foundation," and thus was born ZF. (A few words about Fraenkel: Born in Munich, attending and teaching at a variety of German universities, he was both a devoted mathematician and a Zionist. He left Germany in 1929 and then again in 1933 to join the faculty of the newly founded Hebrew University in Jerusalem.)

Also in 1921, Thoralf Skolem (1887–1963), a Norwegian mathematician, suggested another formulation of the axiom of replacement, and thus was born ZFS (or ZSF, if you prefer). The Axiom of Choice proved to be a hard nut to crack. Henri Poincaré said: No, no. Jacques Hadamard said: Hmm, let's see if we can get around it. I'm not sure of the reasoning of these fellows, but C leads to counterintuitive monstrosities, such as non-measurable sets and the Banach–Tarski paradox, which says, briefly, that you can take a pea, divide it into a finite number of pieces, and reassemble the pieces using translations and rotations into two peas of the same size as the original. Solution to the world food problem? Well, as the Queen said to Alice: "I've believed as many as six impossible things before breakfast." Of course, we've lived with the magic of infinity ever since Galileo set the square numbers into a one-to-one correspondence with the integers.

If C was a hard nut, it nonetheless provided tremendous stimulation and nourishment for set theorists. In 1940 Kurt Gödel showed that C is consistent with ZF, and in 1963 Paul Cohen showed that C is independent of ZF. Thus was born ZF¬C. One of the axioms of ZF is that of "foundation": No set can be an element of itself. But this has been denied, in AFA (the anti-foundation axiom), which gave birth to $ZFCA^{-}$. There is also the GBvN (Gödel–Bernays–von Neumann) formulation of set theory. The designations within this plethora of set theories remind me of the acronyms of the New Deal social programs, popularly known as Roosevelt's "alphabet soup." ZFC appears to be the theory of choice (an old pun) for most pure mathematicians.

Argumentation within a theory depends on a formalization of logic, and we can choose from a salad bar of alternative logics, including, among others, many-valued logics, intuitionistic logic (which denies the law of the excluded middle), infinitary, modal, non-monotonic, temporal, and fuzzy logics, each capable of interacting within an appropriate set theory. And some authors have envisioned a logic, yet to be developed, that embraces probability. Each system has its axioms, modes of deduction, procedures, and consequences; and we are invited to select whichever seems sensible, convenient, useful, beautiful, or digestible. The varieties of modes of human thought cannot be encapsulated and formalized in a few paragraphs.

Can set theory be a foundation for mathematics? Can logic? Quite apart from Gödel's Incompleteness Theorem, which put an end to the aspirations of logic, the answer to both of these questions seems to be: Yes, if you grant the existence of many foundations and hence of many different systems of mathematics. My personal perception is that mathematics can't have axiomatic foundations and, in fact, doesn't need them. My reason, briefly, is that mathematics is vitally embedded in natural languages, such as English. If one agrees with the Chomsky school that the brain is biologically pre-wired for language, the question is no longer one of mathematical symbolisms. In any case, formalized abstractions do not help us explain how we use language to do all the things that we do with it.

Let me return to Zermelo as an individual. He was brilliant, cranky, contentious, combative, and polemical. He did not like cliques or groups. He had spats with practically everyone he met up with scientifically: Boltzmann, Felix Bernstein, Fraenkel, Skolem, Gödel, and Hermann Weyl, to name a few. In 1927, he and Fraenkel proposed to work together on a biography of Cantor; the collaboration broke down in acrimony. (Fraenkel, I've heard tell, was no pussy cat.) Of Gödel, Zermelo said, "I assert publicly that Gödel's much admired 'proof' is nonsense." If you enjoy reading about controversies, you will find detailed accounts to admire here.

In 1933, with the onset of the Nazi regime, the philosopher Martin Heidegger, appointed rector of the University of Freiburg, "ordered that lectures be opened with the Hitler salute." To his great credit, Zermelo balked. In 1935, he was stripped of his honorary professorship at Freiburg (it would be restored at the end of the war).

Ebbinghaus has done a magnificent job in fleshing out Zermelo's life and work. He has included in his book much correspondence and an appendix that contains selected original German versions. A curriculum vitae, many photos, and full bibliographical references are also provided.

I wish to thank Kay O'Halloran, a semioticist and mathematician at the University of Singapore, for a number of perceptive observations on a draft of this review.

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