From *SIAM News*, Volume 38, Number 7, September 2005

Computational Models of Turbulence: The LANS–α Model and the Role of Global Analysis

By Darryl D. Holm and Edriss S. Titi

Over the last 50 years, researchers have proposed numerous computational models of turbulence for obtaining closure. The objective of closure is to capture certain statistical features of the physical phenomenon of turbulence at computably low resolution by mimicking the average effects of the small scales on the larger ones, without calculating the former explicitly.

The Lagrangian-averaged Navier–Stokes–alpha (LANS–α) model (also known in the literature as the viscous Camassa–Holm equations, or the Navier–Stokes–α model) is the first turbulence closure model produced by Lagrangian averaging. The LANS–α model combines Lagrangian-averaged nonlinearity with Navier–Stokes viscosity. In modifying the nonlinearity of the Navier–Stokes equation, instead of its dissipation, LANS–α provides a way to reach closure without enhancing viscosity. Derivation from these first principles implied mathematical theorems for its solutions, thereby guaranteeing that the most basic features and statistical properties of the flow (energy transport, circulation, variability, instability, dissipation anomaly, and intermittency) at length scales above the effective cutoff scale of alpha are all modeled "accurately."

It has been proved analytically that the LANS–α solutions converge to certain solutions of the three-dimensional Navier–Stokes solutions. Moreover, the model's solutions for nonzero alpha possess a global attractor whose fractal and Hausdorff dimensions are finite. The finite number of degrees of freedom in this model guarantees that the solutions are computable in finite resolution. Details can be found in [4], [10–11].

The theorem-based approach of the LANS–α model development raises new mathematical possibilities for the derivation and analysis of other computational models of turbulence. Application of the alpha model is still in its infancy, but results so far suggest that this new approach will complement, and in some cases subsume, earlier approaches for modeling turbulence in real-world applications.

What Do the Navier–Stokes Equations Say about Turbulence?

Turbulence is an outstanding unsolved multiscale problem of classical physics. It occurs spontaneously in a fluid, when forcing by stirring at the large scales is transferred by convection into slender, swirling circulations in the flow. These coherent swirling "blobs" of fluid, pierced by vortex lines and bounded by material circulation loops, are called eddies. The eddies are Lagrangian structures—that is, they travel with the flow, stretching into extended shapes (sheets or tubes) as they follow the flow induced by the vortex lines that pierce them. The coherent eddies, sheets, and tubes of vorticity, stretching into finer and finer shapes, can be thought of as the "sinews" of turbulence.

The characteristic features of a turbulent flow—the distribution of eddy sizes, shapes, speeds, vorticity, circulation, nonlinear convection, and viscous dissipation—can all be captured with the exact Navier–Stokes equations. The Navier–Stokes equations correctly predict how the cascade of turbulent kinetic energy and vorticity accelerates and how the sinews of turbulence stretch out to finer and finer scales, until their motions reach scales of only a few molecular mean free paths, where they can finally be dissipated by viscosity into heat. Further details can be found in [9], [12–13], [28]. The fidelity of the Navier–Stokes equations in capturing the cascade of turbulence, however, is also their downfall for direct numerical simulations of turbulence.

The Need for a Computational Turbulence Model

Turbulence is a paradigm for nonlinear phenomena. Indeed, without the nonlinear term representing convection in the Navier–Stokes equations, the dynamics of the time-dependent Stokes problem would be trivial. For this reason, the dimensionless Reynolds number Re, a measure of the intensity of the nonlinear inertial term in the Navier–Stokes equations in comparison with the viscous linear effects, is usually used as an indicator of the complexity of a turbulent flow. Heuristic physical arguments attributed to Landau suggest that the number of active degrees of freedom required to simulate the turbulent cascade in high-Reynolds-number flows increases as $Re^{9/4}$. Because turbulent flows often have $Re > 10^6$, simulation requirements quickly outstrip the numerical resolution capabilities of even the largest, most powerful computers.

To make turbulence "computable," then, it is necessary to forsake computing some of the fine details of turbulent flows. To do so without compromising the main statistical features of the physical phenomena, scientists have developed various approximate models that halt the cascade into smaller, faster eddies. Most models accomplish this by causing eddies below a certain size to dissipate computationally into heat. This dissipative imperative causes errors, however, because it damps out the variability (known as "intermittency") in the larger-scale flow that is caused by the myriad motions at small scales interacting nonlinearly in the fields of the larger motion. Thus, computational turbulence closure models based on reducing Reynolds number by enhancing viscous dissipation over its Navier–Stokes value run the risk of producing unrealistically low variability.

Clues from Mathematical Analysis

Remarkably, one of the clues for understanding how turbulence closure models can be developed without enhancing viscous dissipation came from the great analyst Leray. In the first regularization of the Navier–Stokes equations, Leray [22] modified their nonlinearity to the well-known form

$$\frac{\partial}{\partial t}\overline{\mathbf{v}} + \overline{\mathbf{u}}\cdot\nabla\overline{\mathbf{v}} + \nabla\overline{p}$$
$$= \nu\Delta\overline{\mathbf{v}} + \mathbf{F}, \operatorname{div}\overline{\mathbf{v}} = 0,$$

with $\overline{\mathbf{v}} = 0$ on the boundary. Here, ν is the (constant) kinematic viscosity coefficient, $\mathbf{F}$ is the prescribed external force, and $\overline{\mathbf{u}} = G_\alpha * \overline{\mathbf{v}}$ is a filtered version of the regularized velocity $\overline{\mathbf{v}}$. The filtering operation is defined by $G_\alpha * \overline{\mathbf{v}} = \int G_\alpha(\mathbf{x},\mathbf{y})\, \overline{\mathbf{v}}(\mathbf{y})\, d^3y$ for a radially symmetric smooth kernel $G_\alpha(\mathbf{x},\mathbf{y})$ of characteristic width α. The Navier–Stokes equations for $\overline{\mathbf{v}}$ are recovered in the limit as $\alpha \to 0$, so that $\overline{\mathbf{u}} \to \overline{\mathbf{v}}$.

In an insightful review of the Leray regularization of the Navier–Stokes equations [14], G. Galovotti made the point that the Leray regularization no longer satisfies the Kelvin circulation theorem. Galovotti [14] challenged the turbulence community to produce a regularization of the Navier–Stokes equations that does satisfy a Kelvin circulation theorem. Remarkably, the combination of Lagrangian averaging (time-averaging at fixed Lagrangian coordinates) and Taylor's hypothesis (that the fluctuations are of low enough power to be regarded as carried along by the mean flow) leads to the LANS–α model, which produces a regularized equation set that answers Galovotti's challenge. These regularized equations constitute the LANS–α model:

$$\frac{\partial}{\partial t}\overline{\mathbf{v}} + \overline{\mathbf{u}}\cdot\nabla\overline{\mathbf{v}} + \nabla\overline{\mathbf{u}}^T\cdot\overline{\mathbf{v}} - \frac{1}{2}\nabla$$
$$\left(|\overline{\mathbf{u}}|^2 + \alpha^2|\nabla\overline{\mathbf{u}}|^2\right) + \nabla\overline{p}$$
$$= \nu\Delta\overline{\mathbf{v}} + \mathbf{F}, \quad \operatorname{div}\overline{\mathbf{u}} = 0. \qquad (1)$$

Here, α is a constant of length; the filtering relation $\overline{\mathbf{u}} = G\alpha * \overline{\mathbf{v}}$ for the LANS–α model is specified to be $\overline{\mathbf{v}} \equiv \overline{\mathbf{u}} - \alpha^2\Delta\,\overline{\mathbf{u}}$.

Applying Filtering in Kelvin's Circulation Theorem

- The filtering kernel G_α for the LANS–α model turns out to be the Green's function for the Helmholtz operator, $(1 - \alpha^2\Delta)$.
- As stated earlier, the LANS–α motion equation satisfies the Kelvin circulation theorem:

$$\frac{d}{dt}\oint_{c(\overline{\mathbf{u}})}\overline{\mathbf{v}}\cdot d\mathbf{x}$$
$$= \oint_{c(\overline{\mathbf{u}})}\left(\frac{\partial}{\partial t}\overline{\mathbf{v}} + \overline{\mathbf{u}}\cdot\nabla\overline{\mathbf{v}}\cdot\nabla\overline{\mathbf{u}}^T\cdot\overline{\mathbf{v}}\right)\cdot d\mathbf{x}$$
$$= \oint_{c(\overline{\mathbf{u}})}(\nu\Delta\overline{\mathbf{v}} + \mathbf{F})\cdot d\mathbf{x}.$$

- The circulation theorem tells us that the rate of change of momentum per unit mass $\overline{\mathbf{v}}$ around a closed material loop $c(\overline{\mathbf{u}})$ moving with velocity $\overline{\mathbf{u}} = G_\alpha * \overline{\mathbf{v}}$ is given by the integral around that loop of the tangential component of the sum over forces (viscous and external) acting on the fluid.
- This statement of the circulation theorem can also serve as a mnemonic, allowing the derivation of other regularized turbulence models of the LANS–α type simply by specifying a different filtering kernel G_α.

A Brief History of the LANS–α Model

Although we can now write the LANS–α model directly from its circulation theorem, the approach used historically [4] for deriving the closed Eulerian form (1) of the LANS–α motion equation was based on a combination of two earlier results. First, the Lagrangian-averaged variational principle of [16] was applied to derive the inviscid averaged nonlinear fluid equations, obtained by averaging Hamilton's principle for fluids over the rapid phase of their small turbulent circulations at fixed Lagrangian coordinates. (This step had its own precedent in the earlier work on Lagrangian-averaged fluid equations of [1].)

Second, the Euler–Poincaré theory for continuum mechanics of [20] was used to produce the Eulerian form of the equations resulting from the Lagrangian-averaged fluid variational principle. This step determined the relation between the momentum per unit mass $\overline{\mathbf{v}}$, the velocity of the Lagrangian-averaged fluid $\overline{\mathbf{u}}$, and the Lagrangian fluctuation statistics. Next, Taylor's hypothesis of frozen-in turbulence circulations was invoked for closing the Eulerian system of Lagrangian-averaged fluid equations, by obtaining the explicit relation $\overline{\mathbf{v}} \equiv \overline{\mathbf{u}} - \alpha^2\Delta\,\overline{\mathbf{u}}$. Finally, Navier–Stokes Eulerian viscous dissipation was added, so that viscosity would cause diffusion of the newly defined Lagrangian-averaged momentum and monotonic decrease of its total Lagrangian-averaged energy. The theory can be either isotropic or anisotropic. Details of the derivation are provided in [17–18]. An alternative derivation is given in [24].

Turbulence Modeling: A New Role for Leray Analysis

At this point, we can consider the LANS–α model as simply a regularization of the Navier–Stokes equations and re-examine its properties from the viewpoint of Leray's analysis. It turns out, moreover, that the same ideas that restore Kelvin's circulation theorem to Leray's regularization of the Navier–Stokes equations also provide a basis for proposed computational models of turbulence, including large eddy simulations.

The converse is also true: Any proposed model of turbulence will lead to likely candidates for application of Leray's analysis. Consequently, the classical Leray analysis of the Navier–Stokes equations has a new role in the study of the analytical properties of turbulence models. Indeed, *the Leray model itself* was recently found to be a viable candidate for computational modeling of turbulence [8], [15].

Relation of LANS–α Inertial Subrange to Taylor's Turbulence Microscale

Of the various remarkable properties of the LANS–α system of equations (1), we discuss here only one: the t*wo different scalings* of its inertial regime, depending on whether the eddies are larger or smaller than alpha. In fact, the Kármán–Howarth theorem for the LANS–α system, discussed in [19], implies that the *translational* kinetic energy spectrum $E(k)$ (i.e., the energy spectrum of the translational velocity as a function of the wave number k) changes from $E(k) \sim k^{-5/3}$ for large scales, corresponding to wave numbers $k\alpha \ll 1$, to $E(k) \sim k^{-3}$ for small scales, corresponding to wave numbers $k\alpha \gg 1$. A dimensional argument justifying this change of scaling in the inertial regime for the LANS–α model was first given in [10].

With this wave-number scaling, *the inertial range is shortened* for circulations with high wave numbers (i.e., $k\alpha > 1$) in the LANS–α model. With α fixed, the wave number k_α at the end of the second, steeper, k^{-3} regime of the LANS–α inertial range is determined in [10] to be

$$k_\alpha \sim \left(\frac{1}{\alpha}\right)^{1/3} k_{Ko}^{2/3}.$$

This is the wave number k_α at which dissipation balances nonlinearity in turbulence described by the LANS–α equations. In this formula, k_{Ko} is the Kolmogorov dissipation wave number, at which dissipation balances nonlinearity in turbulent solutions of the Navier–Stokes equations. Because k_{Ko} scales with integral-scale Reynolds numbers as $Lk_{Ko} \approx Re^{3/4}$, with L denoting the integral scale (or domain size), we find that dissipation balances nonlinearity for the LANS–α model at $Lk_\alpha \approx Re^{1/2}$. Remarkably, the wave number for the well-known Taylor microscale also scales as $Re^{1/2}$ [27]. Thus, for the three progressively larger wave numbers:

$$L/\alpha < Lk_\alpha \approx Re^{1/2} < Lk_{Ko} \approx Re^{3/4}.$$

Shortening the inertial range for the LANS–α model to $Lk < Lk_\alpha \approx Re^{1/2}$, rather than the $Lk < Lk_{Ko} \approx Re^{3/4}$ of the Navier–Stokes equations, implies fewer active degrees of freedom in the solution for LANS–α, which, as discussed below, makes LANS–α much more computable than Navier–Stokes at high Reynolds numbers.

Counting Degrees of Freedom

For turbulence that is "extensive" in the thermodynamic sense, we might expect the number of "active degrees of freedom" N_{dof} for alpha-model turbulence to scale as

$$N_{dof}^{\alpha} \equiv (Lk_\alpha)^3 \sim (L/\alpha)$$
$$(Lk_{Ko})^2 \sim \frac{L}{\alpha} Re^{3/2},$$

where k_α is the end of the LANS–α inertial range and $Re = L^{4/3}\varepsilon^{1/3}/\nu$ is the integral-scale Reynolds number (with total energy dissipation rate ε and viscosity ν). Because the corresponding number of degrees of freedom for Navier–Stokes with the *same* parameters is

$$N_{dof}^{NS} \equiv (Lk_{Ko})^3 \sim Re^{9/4},$$

a possible trade-off emerges in the relative Reynolds number scaling of the two models, provided resolution down to the Taylor microscale. (In practice, users of the LANS–α model often obtain acceptable results by setting the resolution scale at just half the size of alpha.)

Should these estimates not prove overly optimistic, the implication would be a two-thirds power scaling advantage for use of the LANS–α model. In other words, in needing to resolve only the Taylor microscale, the LANS–α model could compute accurate results (at scales larger than alpha) by using two decades of resolution in situations that would require three decades of resolution for the Navier–Stokes equations, at sufficiently high Re. This is because the number of degrees of freedom for the two models scales as

$$N_{dof}^{\alpha} \equiv (Lk_\alpha)^3 \sim (L/\alpha)(Lk_{Ko})^2$$
$$\sim \frac{L}{\alpha} Re^{3/2} \sim \left(N_{dof}^{NS}\right)^{2/3}.$$

$Re^{3/2}$ Scaling Estimate for the Hausdorff Dimension of the LANS–α Global Attractor

These dimensional arguments were substantiated by a slightly better estimate with the same $Re^{3/2}$ scaling when the fractal and Hausdorff dimensions of the global attractor for the LANS–α model were estimated in [11]. In addition, the well-posedness of the LANS–α model in a bounded domain was confirmed in [23].

Numerical Speed-up of LANS–α over Navier–Stokes

A numerical argument for the speed-up advantage of the LANS–α model in comparison with direct simulations of the Navier–Stokes equations goes as follows. The LANS–α model gains one factor of $(N_{dof}^{NS}/N_{dof}^{\alpha})^{1/3}$ in relative increased computational speed for each spatial dimension and another factor (at least) for the accompanying reduced Courant–Friedrichs–Levy (CFL) time-step restriction. Altogether, the gain in speed would be

$$\left(\frac{N_{dof}^{NS}}{N_{dof}^{\alpha}}\right)^{4/3} \sim \frac{\alpha}{L} Re.$$

Because $\alpha/L \ll 1$ and $Re \gg 1$, the two factors on the right side do compete; the Reynolds number should eventually win out, however, because Re can continue to increase, while α/L is expected to tend to a constant value, say $\alpha/L = 1/100$, at high (but experimentally attainable) Reynolds numbers, at least for simple flow geometries. Empirical indications of this tendency were found in [4–6] in comparisons of steady LANS–α solutions with experimental mean-velocity-profile data for turbulent flows in pipes and channels.

Thus, according to this scaling argument, a factor of 10^4 in increased speed for accurate computation of scales greater than α could be achieved, by using the LANS–α model at the Reynolds number for which $k_{Ko}/k_{\alpha} = 10$. An early indication of the feasibility of obtaining such increases in computational speed was realized in the direct numerical simulations of homogeneous turbulence reported in [7], in which $k_{Ko}/k_{\alpha} \simeq 4$ and the full factor of $4^4 = 256$ in computational speed was obtained with spectral methods in a periodic domain at little or no cost of accuracy in the statistics of the re-solved degrees of freedom, i.e., those with $k\alpha < 1$.

Outlook

Further steps are being taken at Los Alamos National Laboratory, the National Center for Atmospheric Research, and elsewhere to test whether the LANS–α model will continue to live up to its promise for fast accurate numerical simulations of turbulence when additional physical processes are included in these computations. At Los Alamos, the LANS–α model has been extended to include rotation, topography, and buoyancy stratification for applications in ocean and atmosphere circulation studies for global climate modeling. At NCAR, the LANS–α model has been extended to include magnetic fields, so that researchers there now stand at the threshold of being able to model the effects of turbulence on the dynamics of the geodynamo and the solar dynamo.

Several other variants of the LANS–α model have also been investigated analytically and numerically for incompressible turbulence, e.g., in pipes and channels. The steady solutions of all of these variants compare well with the measurements of mean velocity in turbulent flows in pipes and channels over a wide range of Reynolds numbers and for a constant value of alpha that is small (about one percent of the pipe diameter or channel width). Numerical results for two of the primary variants, the Leray–α model [8], [15] and the Clark–α model [2], are promising; analytical estimates prove that everything said here about the computability, energy spectrum, and finite-dimensional global attractor for the LANS–α model also holds true for these two alternative models. The characteristic preservation of the Kelvin circulation theorem for Navier–Stokes of the LANS–α model, however, is not a feature of the Leray–α and Clark–α models.

The challenge first enunciated in [14] of developing a regularization of the Navier–Stokes equations that preserves Kelvin's circulation theorem was answered (accidentally) in the development of the LANS–α model. After a promising beginning, the eventual roles of both Kelvin's circulation theorem and the global analysis of PDEs in developing and analyzing computational turbulence models remain to be fully determined.

Acknowledgments

The authors are grateful to friends and colleagues for their comments, encouragement, and constructive suggestions, especially the Turbulence Working Group at Los Alamos and its advisory committee (S.Y. Chen, A.J. Domaradzky, R. Donnelly, G. Eyink, U. Frisch, R.M. Kerr, A. Pouquet, and S.R. Sreenivasan), as well as the participants in the group's annual turbulence workshops during the past four years.

The work described in this article was supported by the U.S. Department of Energy (by contract W–7405–ENG–36 and by the AMR/MICS program of the Office of Science) and the National Science Foun-dation. The work of E.S.T. was supported in part by NSF (DMS–0204794) and by the U.S. Civilian Research and Development Foundation (RM1–2343–MO-02).

References

[1] D.G. Andrews and M.E. McIntyre, *An exact theory of nonlinear waves on a Lagrangian-mean flow*, J. Fluid Mech., 89 (1978), 609–646.

[2] C. Cao, D.D. Holm, and E.S. Titi, *On the Clark–α model of turbulence: global regularity and long-time dynamics*, submitted; http://arxiv.org/abs/nlin.CD/0412007.

[3] C. Cao and E.S. Titi, *Analytical study of the long-term dynamics of the Smagorinksy model of turbulence*, submitted.

[4] S. Chen, C. Foias, D.D. Holm, E.J. Olson, E.S. Titi, and S. Wynne, *The Camassa–Holm equations as a closure model for turbulent channel and pipe flows*, Phys. Rev. Lett., 81 (1998), 5338–5341.

[5] S. Chen, C. Foias, D.D. Holm, E.J. Olson, E.S. Titi, and S. Wynne, *A connection between the Camassa–Holm equations and turbulence in pipes and channels*, Phys. Fluids, 11 (1999), 2343–2353.

[6] S. Chen, C. Foias, D.D. Holm, E.J. Olson, E.S. Titi, and S. Wynne, *The Camassa–Holm equations and turbulence in pipes and channels*, Physica D, 133 (1999), 49–65.

[7] S.Y. Chen, D.D. Holm, L.G. Margolin, and R. Zhang, *Direct numerical simulations of the Navier–Stokes alpha model*, Physica D, 133 (1999), 66–83.

[8] A. Cheskidov, D.D. Holm, E.J. Olson, and E.S. Titi, *On a Leray–α model of turbulence*, Proc. Roy. Soc. London A, to appear, 2005.

[9] C. Foias, *What do the Navier–Stokes equations tell us about turbulence?* in *Harmonic Analysis and Nonlinear Differential Equations* (November 3–5, 1995, University of California, Riverside), Contemporary Mathematics: 208, American Mathematical Society, Providence, 1997, 151–180.

[10] C. Foias, D.D. Holm, and E.S. Titi, *The Navier–Stokes–alpha model of fluid turbulence*, Physica D, 152 (2001), 505–519.

[11] C. Foias, D.D. Holm, and E.S. Titi, *The three dimensional viscous Camassa–Holm equations, and their relation to the Navier–Stokes equations and turbulence theory*, J. Dyn. and Diff. Eqns., 14 (2002), 1–35; http://xxx.lanl.gov/abs/nlin.CD/0103039.

[12] C. Foias and R. Temam, *The connection between the Navier–Stokes equations, dynamical systems and turbulence theory*, in *Directions in Partial Differential Equations*, M.C. Randall, R.H. Rabinowitz, and M.E. Turner, eds., Academic Press, Boston, 1987, 55–73.

[13] U. Frisch, *Turbulence: The Legacy of A.N. Kolmogorov*, Cambridge University Press, Cambridge, UK, 1995.

[14] G. Galovotti, *Some rigorous results about 3D Navier–Stokes*, in *Les Houches 1992 NATO ASI Meeting on Turbulence in Extended Systems*, R. Benzi, C. Basdevant, and S. Ciliberto, eds., Nova Science, New York, 1993, 45–81.

[15] B.J. Geurts and D.D. Holm, *Regularization modeling for large-eddy simulation*, Phys. Fluids, 15 (2003), L13–L16.

[16] I. Gjaja and D.D. Holm, *Self-consistent wave-mean flow interaction dynamics and its Hamiltonian formulation for a rotating stratified incompressible fluid*, Physica D, 98 (1996), 343–378.

[17] D.D. Holm, *Fluctuation effects on 3D Lagrangian mean and Eulerian mean fluid motion*, Physica D, 133 (1999), 215–269.

[18] D.D. Holm, *Lagrangian averages, averaged Lagrangians, and the mean effects of fluctuations in fluid dynamics*, Chaos, 12 (2002), 518–530.

[19] D.D. Holm, *Kármán–Howarth Theorem for the Lagrangian averaged Navier–Stokes alpha (LANS–α) model*, J. Fluid Mech., 467 (2002), 205–214.

[20] D.D. Holm, J.E. Marsden, and T.S. Ratiu, *The Euler–Poincaré equations and semidirect products with applications to continuum theories*, Adv. in Math., 137 (1998), 1–81.

[21] A.N. Kolmogorov, *Dissipation of energy in a locally isotropic turbulence*, Dokl. Akad. Nauk SSSR, 32 (1941), 141–143. (English translation in Proc. Roy. Soc. London A, 434 (1991), 15–17.)

[22] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math., 63 (1934), 193–248.

[23] J.E. Marsden and S. Shkoller, *Global well-posedness for the Lagrangian averaged Navier–Stokes (LANS–alpha) equations on bounded domains*, Philos. Trans. Roy. Soc. London A, 359 (2001), 1449.

[24] J.E. Marsden and S. Shkoller, *The anisotropic Lagrangian averaged Euler and Navier–Stokes equations*, Arch. Rational Mech. Anal., 166 (2003), 27–46.

[25] L.F. Richardson, *Atmospheric diffusion shown on a distance-neighbour graph*, Proc. Roy. Soc. London A, 110 (1926).

[26] G.I. Taylor, *Diffusion by continuous movements*, Proc. London Math. Soc., 20 (1921), 196–212.

[27] G.I. Taylor, *The spectrum of turbulence*, Proc. Roy. Soc. London A, 164 (1938), 476–490.

[28] M.I. Vishik and A.V. Fursikov, *Mathematical Problems of Statistical Hydrodynamics*, Kluwer Academic Press, Dordrecht, 1980, 709–737.

Darryl D. Holm (dholm@lanl.gov) is a researcher in the Computational and Computer Science Division at Los Alamos National Laboratory and a professor of mathematics at Imperial College, London. Edriss S. Titi (etiti@math.uci.edu) is a member of the Departments of Mathematics and of Mechanical and Aerospace Engineering at the University of California, Irvine, and the Department of Computer Science and Applied Mathematics at the Weizmann Institute of Science.